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A Theory of Truthmaker Content I: Conjunction, Disjunction and Negation

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Abstract I develop a basic theory of content within the framework of truthmaker semantics and, in the second part, consider some of the applications to subject matter, common content, logical subtraction and ground.

Keywords Semantics · Content · Entailment · Truthmaker · Ground · Negation

The semantic content of a statement is often taken to be its truth-conditional content, as constituted by the conditions under which it is true. But there are somewhat different ways to understand what these truth-conditions are. On the clausal approach, especially associated with the name of Davidson, the truth-conditions of a statement are not entities as such but the clauses by which a truth-theory specifies when a statement is true. On the objectual approach, by contrast, the truth-conditions are objects, rather than clauses, which stand in a relation of truth-making to the statements they make true.[1]

Under the most familiar version of the objectual approach, the truth-conditions of a statement are taken to be possible worlds and the content of a statement may,

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accordingly, be identified with the set of possible worlds in which it is true. Under a somewhat less familiar version of the object-based approach, the truth-conditions are not - or not, in general - possible worlds but possible states or situations - fact-like entities that make up a world rather than the worlds themselves; and the content of a statement may, in this case, be identified with the set of verifying states or situations in which it is true.

In this paper I pursue the last of these options. However, my understanding of what it is for a state to verify - or be a truthmaker for - a statement is somewhat unusual. It is often supposed that verification is monotonic; if a state verifies a statement then so does any more comprehensive state. But on the account that I wish to adopt, this is not generally so. For it is to be a general requirement on verification that a verifier should be relevant as a whole to the statement that it verifies; and in extending a verifier with additional material, this holistic relevance of the verifier to the statement may be lost.

The present approach is still extensional rather than structural - content is given by a set of verifiers rather than by a structured proposition (as found, for example, in Soames [24]); but it is informed by a more refined conception of what the verifiers are and of what it is for them to verify. It is my view that it is only be adopting such an approach that the full potential of the truth-conditional conception of content can be realized. Thus the paper is an implicit argument both against the possible worlds approach, with its inadequate conception of the verifiers of a statement, and against the more usual situation-theoretic approach, with its inadequate conception of verification. I call the new notion of content 'truthmaker content' to distinguish it from the more familiar notion of 'truth-conditional content', defined in terms of possible worlds, although my view is that it is more deserving of the appellation.

The main advantage of the present approach, to my mind, is that it enables us to define a battery of important concepts concerning content that would otherwise be unavailable to us. We can provide an analysis of some of these concepts under a structural conception of propositions and some kind of analysis of these concepts under alternative extensional approaches, but not one which properly matches up with our intuitive ideas or which properly does the work that we want them to do. The present approach therefore provides us with a highly satisfying intermediate position, combining many of the general advantages of the more fine-grained structural approach with the elegance and simplicity of the extensional approach.

The paper is in two parts - the present part dealing with the familiar concepts of conjunction, disjunction and negation and the subsequent part dealing with the less familiar concepts of subject-matter, common content, logical remainder and ground. We shall provide an account of the quasi-structural notions of conjunctive and disjunctive part in the present part, but it is only in the second part that the approach will come into its own and its distinctive contribution to the theory of content become most apparent. Each of the two parts begins with an informal exposition of the material and concludes with a technical addendum. In principle, the exposition and addendum could be read independently of the other, though the reader may find it helpful to go back and forth between them.

The paper is principally concerned with developing the *core theory* of truthmaker content; and it is not at all concerned with applications. I believe that the theory has significant bearing on a wide variety of different topics and fields - including the analysis of verisimilitude within the philosophy of science, the semantics and logic of deontic and imperative statements within philosophical logic, and the treatment of scalar implicature and presupposition within linguistics; and, indeed, it was some of these applications that helped motivate the development of the theory in the first place. But it has not been my concern to deal with them here.

There are a number of points of contact with other recent lines of research, which I have not discussed but which are well worth exploring: something like the clauses for the conjunction, disjunction and negation is to be found in van Fraassen [25], in Schubert [23], and in some subsequent work; the defining clause for partial content is essentially the same as that for the power order in algebra [3] or the Egli-Milner order in computer science; the work of Yablo [26] and Gemes [15, 16] provides related accounts of partial content; and there are a number of intriguing connections with linear logic [17], the theory of bilattices [11–13], and inquisitive semantics [4]. It was my attempt to provide a semantics for Angell's system of analytic entailment (in Fine [7]) that led to the present theory; and the seminal work of Angell [1] may, with some justice, be regarded as the foundation for the present approach.

There are a number of obvious technical questions arising from the present work which I also have not discussed. How might an analogous algebraic theory be developed (the counterpart, within the present framework, to the theory of Boolean algebras within classical logic). How should the account be developed under weaker assumptions on the domain of propositions? How should the account be extended to the quantifiers? I hope it is fairly evident how these questions might be approached even if it is not always exactly clear how they are to be solved.

1 Verification

Central to our theory is the concept of verification or truthmaking. What gets verified is a statement or proposition; and what verifies is a state or situation. The term 'state', for me, is a term of art and it is used to cover not just states in the ordinary sense of the word but facts, events, conditions or whatever else may legitimately be regarded as a truthmaker.

We allow possible (or consistent) states - such as the state of being hot, given that I am cold; and we allow impossible (or inconsistent) states - such as the state of my being both hot and cold.[2] States are taken to stand in mereological relationships to one another - as with the state of my being cold, for example, being part of the state of my being cold and hungry; and it is assumed that any states $s_1, s_2, s_3, \ldots$ will have a fusion $s_1 \sqcup s_2 \sqcup s_3 \sqcup \ldots$, which has all of the states $s_1, s_2, s_3, \ldots$ as parts and is a part of any state that has all of the states $s_1, s_2, s_3, \ldots$ as parts. Thus given that there are states of my being cold and of my being hungry, there will be a fused state of my being both cold and hungry. There will, in particular, be a *full* state $\blacksquare$, which is the fusion of all states and of which all states are therefore a part, and also a *null* state $\square$, which is the fusion of no states and which is therefore a part of every state. The

[2] Fine [8] provides an account of how impossible states may be constructed from possible states.

null state corresponds to a state of ‘mere nothingness’, in which nothing whatever is required for the state to obtain.

Verification is to be exact; when a state verifies a proposition, it must be relevant *as a whole* to the truth of the statement. Thus there is no guarantee that any extension of a verifier for a proposition, i.e. any state that has the verifier as a part, will also be a verifier for the proposition. We do not attempt to define the notion of exact verification - as a minimal verifier, for example. Its nature is taken to be revealed by the nature of the constraints imposed upon it rather than by a definition in other terms

Once we allow states to verify a proposition, we should also allow them to falsify a proposition; and falsification, like verification, will be exact. It might be thought that falsification could be defined in terms of verification, since for a state to falsify a proposition is for it to verify the negation of the proposition. But, within the present framework, it is not clear which set of verifiers should be taken to correspond to the negation of a proposition; and so we treat falsification as an additional primitive, on a par with verification.

2 Propositions

A proposition, at its simplest, may be identified with the set of its verifiers. Thus propositions with the same verifiers are taken to be the same. This is as it is with the possible worlds semantics, but with a more refined conception of the verifiers and of what it is for them to verify.

There are a number of conditions which we may wish to impose on the verifiers of a proposition, the three principal ones being:

Verifiability Every proposition is verifiable, i.e. has a verifier;

Closure The fusion $s_1 \sqcup s_2 \sqcup s_3 \sqcup ....$ of some verifiers $s_1, s_2, s_3, ...$ of a proposition is also a verifier of the proposition;

Convexity Any state t lying between two verifiers s and u of a proposition (with s part of t and t part of u) is also a verifier of the proposition.

It is important to note that ‘verifier’ does not mean ‘actual verifier’. Thus a proposition may have a verifier without being true. It is also important to note that ‘verifier’ does not mean ‘consistent verifier’. Thus a proposition may have a verifier and not be consistent. Indeed, it will turn out that Verifiability is compatible with there being a wide range of different inconsistent propositions.

We obtain different conceptions of exact verification and different domains of propositions depending upon which of these various conditions we accept. If we accept Closure and Convexity, we obtain what I call a domain of *regular* propositions (or a *regular* domain). Regular propositions have an especially simple form. For each such proposition P (if non-empty) will have a maximal verifier $\boldsymbol{p}$, the fusion of all its verifiers, which we identify with its subject-matter - the agglomeration of the facts, so to speak, from which its verifiers are drawn; and it will also have various low lying verifiers, with all other verifiers lying above them. The proposition itself will then consist of all the states that lie between the low lying verifiers and the maximal verifier. Regular propositions are therefore subject to a limited form of monotonicity; given that a state verifies a regular proposition then so does any extension of the state as long as it lies within the subject-matter of the proposition. Thus what makes for the relevance of the verifier is its being suitably 'bound' from above and below; the verifying state must be 'big enough', i.e. bound from below, and not 'too big', i.e. bound from above (something which I have elsewhere called the 'Goldilock's Principle').

Regular propositions give rise to an especially simple and elegant theory of content; and it is my view that the domain of regular propositions is especially suited to certain 'scientific' applications - such as belief revision, confirmation or verisimilitude - for which a highly fine-grained theory of content is not required. However, for a number of other applications, especially within linguistics, these conditions are too strong and need to be relaxed.

When we give up Closure but not Convexity, we obtain what I call a *semi-regular* domain of propositions; and when we drop both conditions, we obtain the *full* domain of propositions. Verifiability is optional in each of these cases; and there are reasons both for and against allowing the empty set of verifiers to be a proposition. In what follows I shall focus mainly on the case of regular propositions, though there is considerable interest in extending the theory to semi-regular and full domains and also to domains whose propositions are only required to be closed (as in Correia [5]).

The present conception of propositions is *unilateral*; a proposition is simply identified with the set of its verifiers. But we may also adopt a *bilateral* conception of propositions, under which a proposition is identified with a pair of sets of states, the first consisting of its verifiers and the second of its falsifiers. Thus each proposition, under the bilateral conception, will have both a *positive* content, as given by its verifiers, and a *negative* content, as given by its falsifiers. There is, of course, no need to equip a proposition with both kinds of content, if any proposition which agrees on the one will agree on the other. But, as we shall see, there are plausible cases in which this is not so and for which the bilateral conception is therefore required.

Once we allow a proposition to have both positive and negative content, there should be some constraints on how the two contents are related; one should not allow an arbitrary set of verifiers to be paired with an arbitrary set of falsifiers. One source of constraint comes from the modal connections between the verifiers and the falsifiers (we shall later deal with constraints that arise from considerations of relevance).

Let us distinguish between *possible* (or *consistent*) and *impossible* (or *inconsistent*) states. Given such a distinction, we may say that two states s and t are *compatible* if their fusion $s \sqcup t$ is a possible state. We might then impose the following two constraints on the 'modal profile' of any bilateral proposition $\boldsymbol{P} = (P, P')$:

Exclusivity No verifier is compatible with a falsifier (i.e. no member of P is compatible with a member of P');

Exhaustivity Any possible state is compatible with a verifier or with a falsifier (i.e., each possible state is compatible with a member of P or with a member of P').

The first constraint rules out there being too many falsifiers for a given set of verifiers and corresponds to the principle that no proposition should be both true and false; and the second rules out there being too few falsifiers for a given set of verifiers and corresponds to the principle that every proposition should be either true or false. Indeed, let us define a *world* or *world-state* w to be a consistent state that contains any state compatible with it; and let us take a proposition $\boldsymbol{P} = (P, P')$ to be *true* (*false*) *at* a world w if w contains a verifier (respectively, a falsifier) of $\boldsymbol{P}$. Then, under the supposition that the proposition $\boldsymbol{P}$ satisfies Exclusivity, we may show:

No Glut $\boldsymbol{P}$ is not both true and false at any given world;

and, under the supposition that the proposition $\boldsymbol{P}$ satisfies Exhaustivity, we may show:

No Gap $\boldsymbol{P}$ is either true or false at any given world.

Thus the two constraints together correspond to the classical requirement that the proposition $\boldsymbol{P}$ be bivalent. However, in stating these requirements, we do not need to presuppose that there are any world-states with respect to which a proposition is either true or false.

Just as there are two extremal states - the null state $\square$ and the full state $\blacksquare$, there are four extremal unilateral propositions - $T_\square$, $F_\square$, $T_\blacksquare$ and $F_\blacksquare$ - defined as follows:

$$T_\square = \{\square\}$$
$$F_\square = \emptyset$$
$$T_\blacksquare = S$$
$$F_\blacksquare = \{\blacksquare\}.$$

Thus $T_\square$ will be trivially true because nothing (or, more exactly, nothing beyond the state of mere nothingness) is required for it to obtain, while $T_\blacksquare$ is trivially true because it will be verified by any state whatever. Similarly, $F_\square$ will be trivially false since nothing can verify it and $F_\blacksquare$ will be trivially false since it can only be verified by the impossible state $\blacksquare$. Two of these propositions - $T_\square$ and $F_\square$ - have the null state $\square$ as their subject-matter; and two of the propositions - $T_\blacksquare$ and $F_\blacksquare$ - have the full state $\blacksquare$ as their subject-matter. We therefore have the following classification of the four propositions:

	null	full
truth	$T_\square$	$T_\blacksquare$
falsity	$F_\square$	$F_\blacksquare$

When we move to the bilateral sphere, there will be a natural pairing of each these unilateral propositions with a set of falsifiers. In this case, $\boldsymbol{T}_\square$, $\boldsymbol{F}_\square$, $\boldsymbol{T}_\blacksquare$ and $\boldsymbol{F}_\blacksquare$ may be defined by:

$$\boldsymbol{T}_\square = \{T_\square, F_\square)$$
$$\boldsymbol{F}_\square = \{F_\square, T_\square)$$
$$\boldsymbol{T}_\blacksquare = \{T_\blacksquare, F_\blacksquare)$$
$$\boldsymbol{F}_\blacksquare = \{F_\blacksquare, T_\blacksquare).$$

Thus each truth or falsehood is paired with a proposition which has the opposite truth-value but the same subject-matter. $\boldsymbol{T}_{\square}$, for example, will be the proposition that is verified by the null state and falsified by no states, while $T_{\blacksquare}$ will be the proposition that is verified by any state and falsified by the full state $\blacksquare$.[3]

It turns out that $T_{\square}$ and $T_{\blacksquare}$ are necessary truths and that $F_{\square}$ and $F_{\blacksquare}$ are necessary falsehoods. But it should not be thought that $T_{\square}$ and $T_{\blacksquare}$ (or $\boldsymbol{T}_{\square}$ and $\boldsymbol{T}_{\blacksquare}$) are the only necessary truths or that $F_{\square}$ and $F_{\blacksquare}$ (or $\boldsymbol{F}_{\square}$ and $\boldsymbol{F}_{\blacksquare}$) are the only necessary falsehoods. Any proposition which includes $\square$ among its verifiers will be a necessary truth and, even when $\square$ is not one of the verifiers of a proposition, it may still be a necessary truth simply because there is no possible world in which one of its verifying states does not obtain; and similarly in the case of necessary falsehood. In this way, the structure of verification and falsification for necessary truths and falsehoods can be as rich and varied as it is for the contingent truths.

3 The Boolean Operations

By the Boolean operations I mean the operations of conjunction, disjunction and negation - denoted, respectively, by $\wedge$, $\vee$ and $\neg$. I take these to be operations on *propositions*, with the negation operation taking a single proposition into its negation and with the conjunctive and disjunctive operations taking two propositions (or, more generally, any number of propositions) into their conjunction or their disjunction.

Corresponding to the Boolean operations on propositions are the Boolean operators (or connectives) on sentences; and without, I hope, any risk of confusion, we may use the same symbols - $\wedge$, $\vee$ and $\neg$ - for both. Once the Boolean operations are defined, we can straightforwardly employ them in providing a semantics for a language containing the Boolean operators. For the semantic value of each simple sentence of the language can be taken to be a proposition, and the semantic value of the sentence ¬A can be taken to be the negation of the semantic value of A, and the semantic value of A ∧ B (or of A ∨ B) can be taken to be the conjunction (resp. the disjunction) of the semantic values of A and B. I have, for the most part, focused on the abstract realm of content, although everything - or almost everything - of what I say will have a parallel within the less abstract realm of semantics.

Given two unilateral propositions P and Q, then when does a state verify their conjunction $P \wedge Q$? A natural answer, given that verification is exact, and perhaps the only reasonable answer, is:

(1) a state s will verify $P \wedge Q$ iff it is the fusion $p \sqcup q$ of states p and q which respectively verify P and Q.

Or, under a conception of propositions in which they are identified with their sets of verifiers:

[1] $P \wedge Q = \{p \sqcup q : p \in P \text{ and } q \in Q\}$;

[3]For certain purposes, there may also be interest in the 'mixed' cases: $\{T_{\square}, F_{\blacksquare}\}$ and $\{T_{\blacksquare}, F_{\square}\}$.

and similarly when any number of propositions are conjoined. Thus the verifiers of a conjunction are obtained by fusing the verifiers of its conjuncts.

Given two propositions P and Q, then when does a state verify their disjunction $P \vee Q$? A natural answer, given that verification is exact, is:

(2) a state s will verify $P \vee Q$ iff s verifies P or s verifies Q.

Or under a conception of propositions in which they are identified with their sets of verifiers:

[2] $P \vee Q = \{s : s \in P \text{ or } s \in Q\} = P \cup Q$,

and similarly when any number of propositions are disjoined. Thus the verifiers of a disjunction are obtained by pooling the verifiers of its disjuncts.

This last definition is fine if we are working within a full domain of propositions, but it will not be satisfactory if we are working within a constrained domain, such as the domains of semi-regular or of regular propositions, since the relevant constraints may not be preserved under 'pooling'. Suppose, for example, that P contains the state p and Q the state q. Then the regular disjunction of P and Q should contain all of the states that lie between p and $p \sqcup q$ or between q and $p \sqcup q$, in addition to p and q themselves. More generally: if we are working within a regular domain, we should take $P \vee Q$ to be the regular closure of $\{s\colon s \in P \text{ or } s \in Q\}$, i.e. the smallest closed and convex set of states to contain $\{s\colon s \in P \text{ or } s \in Q\}$; and if we are working within a semi-regular domain, we should take $P \vee Q$ to be the semi-regular closure of $\{s\colon s \in P \text{ or } s \in Q\}$, i.e. the smallest convex set of states to contain $\{s\colon s \in P \text{ or } s \in Q\}$.

The definition of negation is more problematic, since there is no way to determine simply from the constitution of a proposition P what its negation should be. The obvious way to solve this problem is to work with the bilateral rather than with the unilateral conception of a proposition. We may then say:

(3) s verifies $\neg\boldsymbol{P}$ iff s falsifies $\boldsymbol{P}$, and s falsifies $\neg\boldsymbol{P}$ iff s verifies $\boldsymbol{P}$.

Or, under a conception of bilateral propositions in which each such proposition $\boldsymbol{P}$ is identified with a pair (P, P') of verifier and falsifier sets, we may say:

[3] $\neg(P, P') = (P', P)$,

with the verifiers and falsifiers reversed.

If we follow through on this line of thought, then the previous definitions of conjunction and disjunction should be extended to bilateral propositions. In this case, we may say, in analogy to (1) and (2) that:

(4) s falsifies $\boldsymbol{P} \wedge \boldsymbol{Q}$ iff s falsifies $\boldsymbol{P}$ or s falsifies $\boldsymbol{Q}$ (or belongs to the appropriate closure of such falsifiers), and

(5) s falsifies $\boldsymbol{P} \vee \boldsymbol{Q}$ iff s is the fusion of a falsifier of $\boldsymbol{P}$ and a falsifier of $\boldsymbol{Q}$.

Or given that $\boldsymbol{P} = (P, P')$ and $\boldsymbol{Q} = (Q, Q')$, we may set:

[4] $(P, P') \wedge (Q, Q') = (P \wedge Q, P' \vee Q')$, and

[5] $(P, P') \vee (Q, Q') = (P \vee Q, P' \wedge Q')$,

thereby completing our account of the application of the Boolean operations to bilateral propositions.

If we draw the distinction between possible and impossible states and introduce world-states or worlds, we are then able to derive the classical truth conditions for propositions $\boldsymbol{P}$ and $\boldsymbol{Q}$ that satisfy the Exclusivity and Exhaustivity conditions above. We will have:

$\boldsymbol{P} \wedge \boldsymbol{Q}$ is true at a world iff both $\boldsymbol{P}$ and $\boldsymbol{Q}$ are true at the world;
$\boldsymbol{P} \vee \boldsymbol{Q}$ is true at a world iff either $\boldsymbol{P}$ or $\boldsymbol{Q}$ is true at the world; and
$\neg\boldsymbol{P}$ is true at a world iff $\boldsymbol{P}$ is not true at the world.

Thus truth-at-a-world will behave on the present approach as it does under the possible worlds approach.

It is important to note that, given clauses (1)–(5), two propositions may agree on their verifiers but differ on their falsifiers. For consider the propositions $\boldsymbol{P} \wedge (\boldsymbol{Q} \vee \boldsymbol{R})$ and $(\boldsymbol{P} \wedge \boldsymbol{Q}) \vee (\boldsymbol{P} \wedge \boldsymbol{R})$, where $\boldsymbol{P}$'s sole verifier is p, $\boldsymbol{Q}$'s is q, and $\boldsymbol{R}$'s is r. Then an application of the clauses to the full propositional domain shows that the verifiers of $\boldsymbol{P} \wedge (\boldsymbol{Q} \vee \boldsymbol{R})$ and $(\boldsymbol{P} \wedge \boldsymbol{Q}) \vee (\boldsymbol{P} \wedge \boldsymbol{R})$ alike are $p \sqcup q$ and $p \sqcup r$. Suppose now that $\boldsymbol{P}$'s sole falsifier is P', $\boldsymbol{Q}$'s is q', and $\boldsymbol{R}$'s is r'. Then $p' \sqcup r'$ and $q' \sqcup p'$ will both be falsifiers of $(\boldsymbol{P} \wedge \boldsymbol{Q}) \vee (\boldsymbol{P} \wedge \boldsymbol{R})$ but will not, as a rule, be falsifiers of $\boldsymbol{P} \wedge (\boldsymbol{Q} \vee \boldsymbol{R})$.

The same line of reasoning will work within a semi-regular domain, since the closure of the falsifiers of $\boldsymbol{P} \wedge (\boldsymbol{Q} \vee \boldsymbol{R})$ under convexity will not in general guarantee that they include $p' \sqcup r'$ or $q' \sqcup p'$. However, the reasoning breaks down for regular propositions. For $p' \sqcup q' \sqcup r'$ will be a falsifier of $\boldsymbol{P} \wedge (\boldsymbol{Q} \vee \boldsymbol{R})$ by closure under fusion and, given that $q' \sqcup p'$ lies between the falsifier P' and the falsifier $p' \sqcup q' \sqcup r'$, $q' \sqcup p'$ will also be a falsifier; and similarly for $p' \sqcup r'$. Indeed, it can be shown quite generally in the case of a regular domain that if it is apparent from the logical specification of two propositions that they have the same verifiers, as in the case above, then it will follow that they have the same falsifiers.[4]

But, even in the case of regular propositions, there are quite plausible non-logical cases in which verification and falsification will come apart. Suppose, for example, that r, g and b are the states of a given object being, respectively, red, green or blue; and, for the sake of simplicity, suppose that there are no other colors. Then the sole verifier for the propositions *RED*, that the given object is red, may plausibly be taken to be r while the sole falsifiers of *RED* may plausibly be taken to be g and b and also $g \sqcup b$ (given that the falsifiers should be closed under fusion); and similarly for the propositions *GREEN* and *BLUE*. Thus $\neg RED$ and *GREEN* $\vee$ *BLUE* will have

[4]This result is implicit in the completeness result for Angell's system in Fine [7].

the same verifiers. However, the sole falsifier of $\neg RED$ will be r, the sole verifier of *RED*, while the falsifiers of *GREEN* $\vee$ *BLUE* will also include $r \sqcup g$, $b \sqcup r$ and $r \sqcup g \sqcup b$.

Of course, in this case, the additional falsifiers of *GREEN* $\vee$ *BLUE* are inconsistent states and so an obvious response to the counter-example is to confine one's attention to *consistent* verifiers and falsifiers and only require that the consistent falsifiers be the same when the consistent verifiers are the same. The above counter-example will not then work, but the modified principle will fail on other grounds. For consider the propositions $\boldsymbol{P} \wedge \neg\boldsymbol{P}$ and $\boldsymbol{Q} \wedge \neg\boldsymbol{Q}$. They will have the same consistent verifiers, viz. none, but they will in general differ with respect to their consistent falsifiers, with $\boldsymbol{P} \wedge \neg\boldsymbol{P}$ being falsified by any verifier or falsifier of $\boldsymbol{P}$ and $\boldsymbol{Q} \wedge \neg\boldsymbol{Q}$ being falsified by any verifier or falsifier of $\boldsymbol{Q}$.

The issue calls for a much more extended discussion, but I think it will be found that there is no view which (a) respects clauses [1–5] of the basic theory, (b) is reasonably tolerant with respect to what the verifiers or falsifiers of a proposition might be, and (c) retains the principle that propositions with the same verifiers should have the same falsifiers. Thus, if I am right, there is a very real tension between (a) and (c), given (b), and the present account of the Boolean operations is therefore not really compatible with a unilateral conception of propositional identity.

There is a way in which this result may not appear too surprising, since we are already familiar with cases in which a unilateral conception may fail, either because of truth-value gluts or because of truth-value gaps. Under a 'possible' worlds semantics, for example, we may allow a proposition to fail to be true or false in a world or we may allow it to be both true and false in a world; and, in either of these cases, there will be no general procedure to determine when a proposition is false from when it is true. But the present case is much more radical than either of these other, more familiar, cases, since it arises even when the propositions in question are bivalent. The source of the failure lies not in the presence of gluts or gaps but in the conditions for the presence of truth not being entirely determinative of the conditions for its absence.

But all may not be lost. For there is a limited way in which we may define the negation of a unilateral proposition, as long as we give up the principle of double negation, according to which the negation of the negation of a proposition is that very proposition.[5] For suppose we are given a relation of *exclusion* which tells us when one state is excluded by another state. We may be told, for example, that the state of an object's having a certain color is excluded by the states of its having some other color or that a person's having a certain height is excluded by his having another height.

Exclusion is a form of incompatibility; if s' excludes s then s' is incompatible with s. But two states may be incompatible without either excluding the other, since it is also required that the excluding state should be wholly relevant to the exclusion of the state that it excludes. Socrates' being a Greek philosopher, for example, will not in the relevant sense exclude his being a Roman philosopher, since his being a

[5] An alternative approach is to adopt the intuitionistic semantics of Fine [6].

philosopher plays no role in preventing him from being a Roman philosopher; and similarly, his being a Roman philosopher will not exclude his being a Greek philosopher. His being Greek, on the other hand, will exclude his being Roman.

The notion of exclusion provides us with a further 'relevantist' way of constraining the interaction between verifiers and falsifiers. For suppose we are given a proposition P whose verifiers are $p_1, p_2, \ldots$. We may then take the *exclusionary negation* $\sim P$ *of* P to be verified by the fusion q of any states $q_1, q_2, \ldots$ which respectively exclude the verifiers $p_1, p_2, \ldots$ of P (or by the suitable closure of such fusions). Thus what we require of a verifier $q = q_1 \sqcup q_2 \sqcup \ldots$ of the negative proposition $\sim P$ is that it should 'knock out' the verifiers of P; each of the components $q_1, q_2, \ldots$ of q should play its part in excluding a corresponding verifier $p_1, p_2, \ldots$ of P.

We cannot in general suppose that the set P' of falsifiers of a bilateral proposition $\boldsymbol{P} = (P, P')$ will coincide with the set $\sim P$ of its excluders since, even if this is true of some given propositions of the form $(P, \sim P)$, it may not be true of the Boolean compounds of such propositions. However, under certain plausible assumptions, the set P' of falsifiers will always be a subset of the set $\sim P$ of excluders; every falsifier will be an excluder, thereby restricting the falsifiers to those that are relevant to the exclusion of the verifiers.

One great advantage in employing the technology of exclusion is that it enables us to simplify the semantics for new linguistic constructions. Suppose, for example, that we wish to give a semantics for the counterfactual construction A > C. Within the possible worlds framework, we need merely give an account of when the counterfactual A > C is true in a given world, since it can be taken to be false at a world when it is not true at the world. But it looks as if, within the present framework, we should give an account, not only of when A > C is verified by a given state, but also of when it is falsified by a given state, since the negative content of a bilateral proposition is not in general a function of its positive content. This is a source of considerable embarrassment, since it would appear to allow a kind of ambiguity in the proposition expressed by linguistic constructions which does not genuinely exist. But in these cases, there is a uniform way of determining the negative content Q of the proposition expressed by A > C (or whatever the new construction might be) from its positive content P, since we can simply take Q to be identical to $\sim P$.

4 The Boolean Operators De-Sententialized

We have provided a theory of the Boolean operations in their application to propositions and of the Boolean operators in their application to sentences. But one remarkable aspect of the theory is that there is nothing in the account of the operations or operators which requires that their application be limited in this way.

Consider again clauses (1) and (2) above for a full domain, but stated with reference to sentences rather than propositions:

(1)′ a state s will verify the sentence A ∧ B iff s is the fusion $p \sqcup q$ of states p and q which respectively verify A and B;

(2)′ a state s will verify the sentence A ∨ B iff s verifies A or verifies B.

Suppose now that, instead of talking of sentences, we talk of denoting expressions of a given kind and that, instead of talking of a state verifying a sentence A, we talk of an expression E denoting an individual. These clauses then become:

(1)* the conjunctive expression E $\wedge$ F will denote the individual d iff d is a fusion of individuals e and f respectively denoted by E and F;
(2)* the disjunctive expression E $\vee$ F will denote the individual d iff d is either denoted by E or denoted by F.

Let us illustrate with the case of nominal expressions, such as 'Socrates', 'Socrates and Plato' and 'Socrates or Plato'. Suppose that E is the name 'Socrates', whose sole denotatum is Socrates, and that F is the name 'Plato', whose sole denotatum is Plato. Then, according to these clauses, the sole denotatum of 'Socrates $\wedge$ Plato' will be the fusion Socrates $\sqcup$ Plato, while the denotata of 'Socrates $\vee$ Plato' will be Socrates and Plato; and similarly for more complex conjunctions and disjunctions of names.[6]

The treatment of negation is not so straightforward. Just as we had to introduce a relation of *falsification* to account for the verification of a negative sentence or proposition, we must now introduce a relation of *preclusion* or *anti-denotation* to account for the denotation of negative denoting expressions. Thus instead of the clause:

(3)′ s verifies the sentence ¬A iff s falsifies A, and
s falsifies the sentence ¬A iff s verifies A,

we have:

(3)* the expression ¬E denotes the individual e iff E precludes e, and
the expression ¬E precludes the individual e iff E denotes e;

and similarly in regard to the preclusion clauses for conjunction and disjunction.

Given that 'Socrates' denotes Socrates, '¬Socrates' will preclude Socrates. What '¬Socrates' denotes will depend upon what 'Socrates' precludes. But what does 'Socrates' preclude? One possible view (though not the only one) goes as follows. We suppose that for any two individuals d and e there is another individual, *d without e*, which we denote by $d \neg e$. This is an individual which will be present in a given scenario when d is present and e is absent. Thus $d \neg e$ is not to be confused with the result $d - e$ of *subtracting* e from d; for $d - d$ will be the null individual $\Box$, which is always present, while $d \neg d$ will be a contradictory individual, which is never present.

As a special case of $d \neg e$, there is the individual $\neg e = (\Box \neg e)$, which will be present when e is absent, given that the null individual $\Box$ is automatically present (alternatively, we can define $d \neg e$ as the fusion $d \sqcup \neg e$ of d and $\neg e$). We may now take the negative individual ¬Socrates to be the sole individual precluded by

[6]The individuals to be fused are best regarded as mereological atoms, since otherwise a fusion may not be uniquely about certain individuals.

'Socrates'; and similarly for other names. The sole individual to be denoted by 'Socrates $\wedge$ $\neg$Plato' will then be Socrates without Plato (i.e. Socrates $\neg$Plato) and the two individuals denoted by '(Socrates $\wedge\neg$Plato) $\vee$ (Plato $\wedge\neg$Socrates)' will be Socrates without Plato (Socrates $\neg$ Plato) and Plato without Socrates (i.e. Plato $\neg$ Socrates).

We can in this way construct a semantics for Boolean compounds of nominal expressions or of expressions in general that is exactly parallel to the semantics for Boolean compounds of sentences.

Under the more abstract version of the sentence-oriented theory, we took each sentence to signify a proposition, which we identified with an ordered pair of unilateral propositions, each unilateral proposition being a set of states. A similar account for arbitrary Boolean expressions may also be given, since we can take an arbitrary Boolean expression to signify the ordered pair of the set of objects it denotes along with the set of objects it precludes. But there might appear to be a difference between the two cases. For, intuitively, there is some one thing, a proposition, which a sentence signifies, but is there some one thing that an arbitrary Boolean expression signifies, apart from the particular objects that it denotes or precludes?

It appears not. However, I think it may be argued that there is a natural category of higher order entities which arbitrary Boolean expressions can be taken to signify. For these purposes it may be helpful to think in terms of menus. Here is an all too typical breakfast menu:

Orange Juice
Bacon and Eggs
Cereal and Milk.

It offers three options, the second and third being a combination of two other options. We might also suppose that certain items are explicitly precluded from the menu, as with:

Orange Juice
Bacon and Eggs
Bacon without Eggs
Cereal and Milk
Granola $\times$

Certain options are 'on' the menu and others are explicitly 'off' the menu either as options on their own (Granola) or as part of another option (Bacon without *Eggs*).

Now a menu can serve any kind of purpose - it can be a menu for achieving truth, for example, rather than gastronomic pleasure; and it can concern any kind of object - a number, say, or a state of affairs, rather than an item of food. Moreover, there is a clear sense in which the very same menu may be presented in different languages or with different words in the same language. We thereby arrive at a very abstract conception of a menu, which is not tied to any particular purpose, or to any particular kind of object, or to any particular symbolic representation; and it is menus in this abstract sense that Boolean expressions may in general be taken to signify, with conjunction corresponding to a combination of items and disjunction to a choice between items.

One advantage of this way of thinking is that it enables us to question the identification of a menu with an object of some particular sort and to consider other accounts of what a menu might reasonably be taken to be. Consider the issue of choice. We have identified the choice between options with the *set* of those options. But one might want to distinguish between an option being offered once and its being offered twice (or more than twice). The set-theoretic representation is then no longer adequate and, indeed, the items denoted and precluded by a Boolean expression will no longer be sufficient to determine what it signifies, since they will provide no means to distinguish between an item being offered once and its being offered twice. In such a case, it would be more appropriate to represent a choice by a multi-set rather than by a set of options.

Similarly in the case of order. One might want the order in which the options are presented to matter; and, in this case, it would be more appropriate to represent a choice by a sequence of options, rather than by a multi-set or a set. For present purposes, the identification of a menu with a pair of sets of options, which are themselves fusions of more basic elements, will serve us well. But there is room for a more general theory of the Boolean operations in which we take a more abstract and flexible view of how the options within a menu might be given.[7]

It has often been assumed that the Boolean operators are essentially logical or truth-theoretic in character and that, even though they may apply to expressions other than sentences, their application to these other kinds of expressions is to be explained in terms of their application to sentences. Thus a disjunctive nominal expression such as ‘Jack or Jill’ is to be understood as an expression which distributes disjunctively in subject-position, so that ‘Jack or Jill went up the hill’ is to have the same truth-conditions as the sentence ‘Jack went up the hill or Jill went up the hill’. It might be conceded that there is a use of conjunctive nominal expression, such as ‘Jack and Jill’, which is collective rather than distributive, so that ‘Jack and Jill married’ would not have the same truth-conditions as ‘Jack married and Jill married’. But it would then be thought that this was a special mereological or plural use of the connective, essentially different from its use as a sentential operator, or else a use that could somehow be derived from the sentential use.

Our own view - at least as embodied in the current theory - is fundamentally different. The Boolean operators have equal and equable application to all expressions whatever, no particular application - whether to sentences or names or some other kind of expression - being privileged. Thus far from being of central significance, the application to sentences or sentence-like expressions is merely one application among other.[8] Moreover, the ur-use of ‘and’ is essentially mereological rather than logical in character; ‘and’, whether used to connect nominal or sentential expressions, will signify fusion.

[7] I believe that we can provide a general theory of ground-theoretic content within such a framework - with conjunction and disjunction being the basic operations by which one content is formed from others. But this is a topic for another day.

[8] A qualification may be in order: for we perhaps may allow the application of the Boolean operators to certain expressions, such as predicates, to be explained in terms of their application to sentences.

We may grant that the sentential use of the Boolean operators has a special truth-theoretic role. Thus we will want to insist that the conjunctive sentence A ∧ B is true just in case A and B are true or that the negative sentence ¬A is true just in case A is false. But this truth-theoretic character does not derive from the meaning of the Boolean operators themselves but from special features of their application in this particular case. For Boolean compounds of sentences denote or preclude (or, as we more naturally say, are verified by or falsified by) states; states obtain or fail to obtain; and a sentence will be true just in case it is verified by a state that obtains and false just in case it is falsified by a state that obtains. The above truth-theoretic principles can then be derived.

Suppose, for example, that A ∧ B is true. Then it is verified by a state s that obtains. But from the general meaning of ∧ it follows that this state will be a fusion of a state s_1 that verifies A and a state s_2 that verifies B. But if s obtains then so do s_1 and s_2; and so A and B will both be true. Suppose now that A and B are true. Then some verifier s_1 of A and some verifier s_2 of B will obtain. But then their fusion will obtain; and so again, it follows from the general meaning of ∧ that A ∧ B is true. The above principle for ¬ and all the other truth-theoretic features of the Boolean compounds of sentences can be derived in a similar way. Thus we see that the truth-theoretic character of the truth-functional connectives is entirely incidental to their core meaning and derives from the special connection between their peculiar denotata and the concept of truth.

The present account of the Boolean operators is suggestive of the theory of denoting phrases presented in Russell's 'Principles of Mathematics', sections 59–62.[9] Russell is primarily interested in phrases involving such particles as 'all', 'every', 'any', 'the', 'a', and 'some', but he considers on the way the use of 'and' and 'or' in nominal constructions, as in the cases mentioned above. He makes distinctions I do not make - he distinguishes, for example, between various uses of 'and' and 'or' (constant, variable, propositional, numerical), where I make no such distinctions; and I, in contrast to him, allow nominal expressions to be negated and to stand in relations of negative as well as positive denotation. But the two theories are alike in appealing to disjunctive and conjunctive 'combinations' and in attempting to explain the meaning of disjunctive and conjunctive nominal expressions in terms of such combinations. Thus Russell's early theory of denoting might with some justice be regarded as a precursor to the present account and the present account might, in its turn, serve to illuminate and provide a rigorous basis for Russell's theory.

The present account may also have application to denoting phrases - or quantifier phrases, in general - as Russell originally intended. The phrase 'a man', for example, may be taken to denote and to preclude the same individuals as the disjunction 'Socrates ∨ Plato ∨ ...', where Socrates, Plato, ... are all the men. Thus it will denote each of Socrates, Plato, ... and preclude the fusion ¬Socrates ⊔ ¬Plato ⊔ ..., i.e the complete absence of all men. The extensional meaning of every other quantifier phrase might be represented in such a way but, what is especially nice, is that we now have the means of making distinctions not available under the more conventional

[9] See Geach [14], Parsons [22], $ 1.1.2 of Kaplan [18] and Bostock [2] for discussion of Russell's early views on denoting.

extensional approaches. The phrase 'at least one man', for example, may be taken to denote and to preclude the same individuals as the disjunction 'Socrates $\vee$ Plato $\vee$ (Socrates $\wedge$ Plato) $\vee$ ...' ,where the disjuncts now denote all individual men and all non-empty fusions of men; and so, although the sentences 'a man F's' and 'at least one man F's' will have the same classical truth-conditions, the denotative meaning of the quantifier phrases 'a man' and 'at least one man' will differ and in a way which is relevant to their linguistic behavior (it will account, for example, for the fact that 'a man', in contrast to 'at least one man', may unproblematically serve as an antecedent to the pronoun 'he').

5 Partial Content

The classical theory of propositions is monolithic. There is a single underlying notion of consequence which may be defined, indifferently, as the relationship that holds between the propositions P and Q when $P = P \wedge Q$ or when $Q = P \vee Q$. The conjunction of two propositions will be the greatest lower bound of those propositions with respect to this relation of consequence - the weakest proposition to have each of them as a consequence, while the disjunction of two propositions will be the least upper bound of those propositions with respect to consequence - the strongest proposition to be a consequence of each of them.

The present theory of propositions, by contrast, is undergirded by two distinct relations of consequence. There is, on the one hand, the relation of *disjunctive part* that holds between P and $P \vee Q$ and there is, on the other hand, the relation of *conjunctive part* (or its converse) that holds between P and $P \wedge$ Q. These two relations are assimilated on the classical approach. For P will be the same proposition as $(P \vee Q) \wedge P$, and so a connection of disjunctive part between P and $P \vee Q$ will also be a connection of conjunctive part between $P \vee Q$ and $P = (P \vee Q) \wedge P$. But the two relations are very different on the current approach and play a very different role.

The two forms of consequence correspond to two ways in which a proposition may be weakened. We may, on the one hand, expand the ways in which it can be verified or we may, on the other hand, remove some part or parts from the verifiers it already has - with the first being a kind of disjunctive weakening and the second a kind of conjunctive weakening. Or to put it more generally in terms of menus, we may either add new items to the menu, thereby providing more choice, or remove some parts of the items already on the menu, thereby reducing the choices we already had.

The relation of disjunctive part - or what we shall now call *entailment* - may be defined in a familiar way; the proposition P will entail Q if every verifier of P is a verifier of Q. But it does not behave in a familiar way. For even though P will entail $P \vee Q$, $P \wedge Q$ will not in general entail P, since a verifier for $P \wedge Q$, which is the fusion of verifiers for P and Q, will not in general be a verifier of P.

The disjunction $P \vee Q$ of two propositions P and Q will be the least upper bound of the propositions with respect to entailment: each of P and Q will entail $P \vee Q$; and if P and Q both entail a proposition R then $P \vee Q$ will entail R. The conjunction $P \wedge Q$ of two propositions P and Q, by contrast, will not stand in any straightforward relation of entailment to those propositions.

The converse of the relation of conjunctive part - or what we shall call *containment* - is defined in a less familiar way. The proposition Q will contain the proposition P just in case two conditions are satisfied: first, every verifier of Q should contain a verifier of P; and, second, every verifier of P should be contained in a verifier of Q. If we lay out the states of Q and of P horizontally on a line, then every state of Q, looking down, should 'see' a state of P that it contains and every state of P, looking up, should see a state of Q that contains it.

Despite the unfamiliar form of the definition, it corresponds, I believe, to a familiar notion and one that, in its own way, is every bit as intuitive as the notion of entailment. One entré into the notion is through the concept of *what is said*. For in saying one thing, one may (among other things) say another. Thus in saying that I am an American philosopher, I am saying that I am a philosopher. But in saying that I am a philosopher, I am not saying that I am a philosopher or American. In the one case, the second content is, in a perfectly intuitive sense, part of the first while, in the second case, it is not.

It is very easy for contemporary analytic philosophers, tainted by decades of indoctrination, to fail to recognize the distinction. But here is a way to restore ourselves to a state of pre-doctrinal innocence. In saying that I am an American philosopher, I am saying something that is partly true (since I am a philosopher, or so I hope, though not American); and this is because a true *part* of what I say is that I am a philosopher. But in saying that I am American, I am not saying something that is partly true or, at least, not on account of its being true that I am an American or a philosopher; for even though my being an American or a philosopher is a true consequence of my being a philosopher, it is not a true *part* of the content and so its truth does not count in favor of its being partially true that I am American. Thus among the true consequences of a proposition, some are part of its content, where these count towards its being partly true, and some are not, where these do not count towards its being partly true.

But even if we are willing to make such a distinction, how can we be sure that it is adequately captured by our definition of containment? One way to justify the definition is through considerations of partial truth of a sort to which we previously alluded. For suppose P is a part of Q. Then, surely, if Q is true then it is partly (or wholly) in virtue of the truth of P that Q is true. But this may be cashed out in terms of truth-conditions as follows: if Q is true then it has a verifier and, for Q to be true partly in virtue of P being true, is for that verifier to contain a verifier of P. Thus every verifier of Q must contain a verifier of P.

To justify the second clause, again suppose P is a part of Q. Then, surely, if P is true then Q is partly (or wholly) true in virtue of P being true. But what this means in terms of truth-conditions is that: if P is true then it has a verifier and, for Q to be partly true in virtue of P being true, is for this verifier to be contained in a verifier of Q. Thus every verifier of P must be contained in a verifier of Q.

These considerations provide us with a justification of the necessity of the two clauses. Although I have no argument for their sufficiency, it is plausible - at least under the unilateral conception of propositions - that no further conditions might reasonably be imposed; and we therefore end up with the present definition.

A further justification for the definition is that it delivers the results and plays the theoretical role that we would expect the relation of containment to play. It will, for example, not be generally true that $P \vee Q$ is part of the content P, for among the verifiers of $P \vee Q$ will be the verifiers of Q and there is no reason, in general, why a verifier of Q should be contained in a verifier of P and so the second clause in the definition of partial content will not be satisfied.

We can also see how P will in general be part of the content $P \wedge Q$. For any verifier of $P \wedge Q$ will be a fusion of a verifier of P and a verifier of Q and so will contain a verifier of P; and, given a verifier of P, it will fuse with a verifier of Q to give a verifier of $P \wedge Q$ that contains a verifier of P (granted that Q has a verifier). Thus the two conditions for partial content will be satisfied and P will be part of the content of $P \wedge Q$.

The conjunctive proposition $P \wedge Q$ will also be the greatest lower bound of P and Q with respect to the relation of containment: as already mentioned, it will contain each of P and Q (if they each have a verifier); and, in addition, it will be contained in any proposition that contains each of P and Q.

We should note, in passing, the anomalous behavior of $F_\Box$. This proposition contains no verifiers and so the conjunction $P \wedge F_\Box$ of any proposition P with $F_\Box$ will also contain no verifiers; and so P will not be contained in $P \wedge\ F_\Box$ when P itself has a verifier p, since p will not be contained in any verifier of $P \wedge F_\Box$.

There are a number of ways of dealing with this problem. One is to ban the proposition and insist that each proposition have a verifier; another is to loosen the definition of conjunctive part and allow any proposition P whatever to be a conjunctive part of $F_\Box$, given that $F_\Box = P \wedge F_\Box$. But each of these modifications comes at a price. If we restrict our attention to verifiable propositions then there will not exist a proposition which is a disjunctive part of every other proposition; and if we loosen the definition of conjunctive part then we will lose the principle that if R is part of Q then $P \vee R$ should be part of $P \vee Q$. For let Q be $F_\Box$. Then any proposition R will be part of $F_\Box$; and so, given the principle, $P \vee R$ will be a part of $P \vee F_\Box = P$.

I have found the most satisfactory solution to be one which countenances $F_\Box$ but treats it as a 'dead end' proposition which is not allowed to be conjoined with other propositions. We need not go so far as to deny that the conjunction of $F_\Box$ with any other proposition *exists*. It is just that there is no need to conjoin $F_\Box$ with any other proposition (it just gives us back $F_\Box$); and we get a smoother theory if the application of conjunction is disallowed in this case. It is as if we were to allow the result of dividing a number by 0 to be another number but forbad this new number from entering into arithmetical operations with other numbers, since it would not then behave in a normal way.

As already noted, a conjunction is the greatest lower bound of its conjuncts with respect to relation of classical consequence, while a disjunction is the least upper bound of its disjuncts with respect to that very relation. But within the present framework, a conjunction will be the greatest lower bound of its conjuncts with respect to containment, while a disjunction will be the least upper bound of its disjuncts with respect to entailment, where neither relation of containment nor entailment is identical to the other or to the relation of classical consequence (which actually bears a rather distant relation to the two other relations). Where conjunction and disjunction serve the same logical master within a classical setting, they serve two different logical masters within the present setting.[10]

We turn to the theory of partial content for bilateral propositions. Suppose that $\boldsymbol{P}$ is the proposition (P, P') and $\boldsymbol{Q}$ the proposition (Q, Q'). Then we may naturally take $\boldsymbol{P}$ to be contained in $\boldsymbol{Q}$ just in case (i) P is contained in Q according to the earlier definition of containment and (ii) P' entails Q' according to the earlier definition of entailment (which means that every falsifier of $\boldsymbol{P}$ should be a falsifier of $\boldsymbol{Q}$).

Perhaps the simplest way to justify the necessity of this further condition is as follows. If (P, P') is to be part of (Q, Q') then (P, P') should be a conjunctive part of (Q, Q'), and so the negation (P', P) of (P, P') should be a disjunctive part of the negation (Q', Q) of (Q, Q') and hence P' should entail Q'.

There is a corresponding definition of entailment in the bilateral case: $\boldsymbol{P} = (P, P')$ will entail $\boldsymbol{Q} = (Q, Q')$ just in case (i) P entails Q and (ii) P' is contained in Q'. Thus we see that the separate notions of entailment and containment for unilateral propositions fuse together in their application to bilateral propositions and become different aspects, oriented in one direction or the other, of a single integrated relation.

It is not, of course, true that:

if $\boldsymbol{P}$ entails $\boldsymbol{Q}$ then $\neg\boldsymbol{Q}$ entails $\neg\boldsymbol{P}$

or that:

if $\boldsymbol{P}$ contains $\boldsymbol{Q}$ then $\neg\boldsymbol{Q}$ contains $\neg\boldsymbol{P}$.

But we do have analogues of these results as long as the type of consequence relation is reversed:[11]

if $\boldsymbol{P}$ entails $\boldsymbol{Q}$ then $\neg\boldsymbol{Q}$ contains $\neg\boldsymbol{P}$; and
if $\boldsymbol{P}$ contains $\boldsymbol{Q}$ then $\neg\boldsymbol{Q}$ entails $\neg\boldsymbol{P}$.

We can show, as before, that the disjunction of two bilateral propositions will be their least upper bound with respect to entailment and that the conjunction of two bilateral propositions will be their greatest lower bound with respect to containment. But the two relations are now connected via their interaction with negation. Conjunction and disjunction are still bound to different masters, but to masters that march in step.

6 Alternative Accounts

Such a theoretically and intuitively satisfying account of partial content (and of the other notions that we shall subsequently introduce) is very difficult, if not impossible,

[10] The domain of propositions has the structure of a lattice from a classical point of view and the structure (or something like the structure) of a bilattice from the present point of view.

[11] Mathematically speaking, we have a Galois connection between entailment and containment with respect to negation.

to achieve under alternative approaches. Suppose one subjects verification to Monotonicity and adopts the above account of partial content. One then loses the reason we previously had for denying that the content $P \vee Q$ was part of the content P, for any verifier of Q may well be contained in a verifier of P by adding it to a pre-existing verifier of P; and nor is it clear how to modify the account so as to avoid such a consequence.

If we adopt the possible worlds approach, then the resources for providing a reasonable definition of partial content become even more limited. Perhaps the best we can do in this case is either to say that proposition Q is part of another P just in case it is necessarily entailed by the other ($Q \supseteq P$, under the identification of propositions with sets of possible worlds) or to say that Q is part of P just in case they are necessarily equivalent ($Q = P$).[12] But the former is far too broad and the latter far too narrow.

Of course, if the required resources are not available under the possible worlds approach then one might try to make them available. The most obvious strategy for doing this is to pick out a distinguished set of propositions, those corresponding to the verifying states or perhaps to something like 'lumps of thought' of Kratzer [19, 20]. These propositions will presumably be closed under conjunction, though not necessarily under either negation or disjunction; and one can then take one state s' to be part of another s if it is necessitated by the other (i.e. if $s' \supseteq s$).

We still need to say what it is for a 'state', in this sense, to verify a given proposition; and here I am aware of three main proposals. Under the first, a state s verifies a proposition P if it necessitates the proposition ($s \subseteq P$); under the second, s verifies P if it *minimally* necessitates P, where this means that no proper part of s necessitates P; under the third, s verifies P if it *compactly* necessitates P, where this might roughly be taken to mean that s is composed of parts that are relevant to whether P is true or false.[13]

Given any one of these accounts of verification, we may plug it into our earlier definition (or something else of the sort) and thereby obtain an account of partial content that appeals to no further notion than that of a lumpy proposition.

But none of these latter three accounts is satisfactory. The first satisfies Monotonicity and so is no better than the situation-theoretic account that we previously considered. The other two do not satisfy Monotonicity but suffer from other difficulties. One problem is that they are not able to satisfy the intuitions that gave rise to the distinction between containment and consequence in the first place. For it will not be generally true that P is part of the content of $P \wedge Q$. Indeed, if it were then $P \vee Q$ would be part of $(P \vee Q) \wedge P$ and hence part of the equivalent content P.

[12] Interestingly, the latter is the result we get if we substitute worlds for states in our own proposed definition, given that no world is a proper part of any other world.

[13] The second of these proposals has been championed by Yablo (in [26] et al.), the third by Gemes [15, 16]. Both authors tend to give somewhat syntactic formulations of their views, but we may see the connection with the present, more abstract, approach if we take the states or 'lumpy' propositions to be the closure under conjunction of all the propositions expressed by the atomic sentences of the language and their negations. I give abstract formulations of their accounts in the Appendix.

Perhaps more serious is the fact that minimal or compact verifiers may not always exist: a proposition may be true and yet have no minimal or compact verifier. Consider, for example, the proposition P expressed by the statement $(p_1 \wedge p_2 \wedge p_3 \wedge ...) \vee (p_2 \wedge p_3 \wedge p_4 \wedge ...) \vee ...$, where the atomic statements p_1, p_2, p_3 ... are independent of one another. Then even if the proposition is true, there will be no minimal verifier, since any verifier of $p_n \wedge p_{n+1} \wedge p_{n+2} \wedge ...$ may always be replaced by a verifier of $p_{n+1} \wedge p_{n+2} \wedge p_{n+3} \wedge ...$ and, likewise, there will be no compact verifier, since any one of the verifiers for the atomic statements $p_1, p_2, p_3, ...$ may be dropped from a verifier or falsifier of P. Thus neither of these two accounts will be of general application.

These difficulties all arise from taking the verifiers of a proposition to be a function of its intensional content (as given by the set of possible worlds in which it is true) and they disappear once one adopts an 'autonomous' conception of verification under which intensionally equivalent propositions may differ in their verifiers. One might, of course, enrich the intensional conception of a proposition with a separate account of its verifiers or/and falsifiers. A proposition might then be represented by an ordered pair (P, V) or an ordered triple (P, V, F) where P is an intensional proposition, i.e. a set of possible worlds, and V is a set of suitably matching verifiers and F a set of suitably matching falsifiers and where, for given first component P, the sets V and F of matching verifiers and falsifiers is allowed to vary. But I believe it will then be found that the intensional component does no essential work and that one might just as well have worked from the start with the components V and F without any mention of P. Intensionality might still do some work in defining the states, since each state can be identified with a suitable set of possible worlds. But I believe that it will be found that even this aspect of the account is also entirely incidental to what we want the account to do and that we might just as well drop *all* reference to possible worlds and work from the start with a primitive set of states ordered by part-whole, as in our own account.[14]

Indeed, one remarkable aspect of the present theory of truthmaker content is that possible worlds completely drop out of the picture. Even the distinction between possible and impossible states is of no real significance, since all that really matters is the mereological structure on the states. One might jokingly remark that the possible worlds approach is fine but for two features: the first is that possible worlds are worlds, i.e complete rather than partial; and the second is that they are possible. Drop both requirements, impose a mereological structure on the resulting states, and we obtain a framework that is of much more help in developing an adequate theory of content.

There is perhaps some point, once we have an account of partial content in these terms, in seeing how it might be 'intensionalized'. To what extent can we say the kind of things we want to say about partial content while being blind to the difference between intensionally inequivalent propositions? I have made a start on such a project towards the end of the Appendix, replacing a proposition, conceived as a single set of verifiers, with a family of such sets. However, the resulting theory is much

[14] This point is further developed in my review (Fine [9] of Yablo [26]).

less satisfactory and natural than the corresponding hyperintensional theory and the mangled objects we end up with in pursuing such a project should not be regarded as a good basis for reconstructing the un-mangled objects with which we began. Instead of starting with an intensional account of propositions and trying as best one can to convert it into a satisfactory hyperintensional account, one should start with a hyperintensional account and then see how best it might be accommodated within an intensional framework.[15]

Appendix: Formal Appendix

Preliminaries

Recall that $\sqsubseteq$ is a *partial order* (po) *on* the set S if it is a reflexive, transitive and anti-symmetric relation on S. Given a po $\sqsubseteq$ on S, we shall make use of the following (mostly standard) definitions. Where $s, t, u \in S$ and $T \subseteq S$:

> s is an *upper bound of* T if $t \sqsubseteq s$ for each $t \in T$;
> s is a *least upper bound* (lub) *of* T if s is an upper bound of T and $s \sqsubseteq s'$ for any upper bound s' of T;
> s is *null* if $s \sqsubseteq s'$ for each $s' \in S$ and other wise is *non-null*;
> s is *full* if $s' \sqsubseteq s$ for each $s' \in S$;
> $s \sqsubset t$ (s is a *proper part of* t) if $s \sqsubseteq t$ but not $t \sqsubseteq s$;
> s *overlaps* t if for some non-null u, $u \sqsubseteq s$ and $u \sqsubseteq t$;
> s is *disjoint from* t if s does not overlap t.

An (*unmodalized*) *state space* $\boldsymbol{S}$ is a pair $(S, \sqsubseteq)$, where S (*states*) is a non-empty set and $\sqsubseteq$ is a binary relation on S subject to the following two conditions:

Partial Order (PO) $\sqsubseteq$ is a po on S;

Completeness Any subset of S has a least upper bound.

The least upper bound of $T \subseteq S$ is unique (since if s and s' are least upper bounds, then $s \sqsubseteq s'$ and $s' \sqsubseteq s$ and so, by anti-symmetry, $s = s'$). We denote it by $\bigsqcup T$ and call it the *fusion of* T (or of the members of T). When $T = \{t_1, t_2, ...\}$, we shall often write $\bigsqcup T$ more perspicuously as $t_1 \sqcup t_2 \sqcup ...$. $\bigsqcup \emptyset$, which we denote by $\square$, is the bottom element of the space and $\bigsqcup S$, which we denote by $\blacksquare$, is the top element. Given that the least upper bound $\bigsqcup T$ always exists, it follows that the greatest lower bound $\bigsqcup T$ will always exist (defined by $\bigsqcup\{s\text{: for all } t \in T, s \sqsubseteq t\}$).

A space $\boldsymbol{S}$ is said to be *distributive* if $s \sqcap (t_1 \sqcup t_2 \sqcup ...) = (s \sqcap t_1) \sqcup (s \sqcap t_2) \sqcup ...$ for any $s, t_1, t_2, ... \in S$. In what follows, we shall assume that the space is distributive, although many of our results will not depend upon this assumption.

In a distributive space, the following principle will hold:

[15] However, as I have argued in Fine [9], it may still be necessary, for certain purposes, to include worlds among the verifying states.

Overlap If s overlaps $t_1 \sqcup t_2 \sqcup$ ... then it overlaps some t_i.

For if s overlaps $t_1 \sqcup t_2 \sqcup$ ..., then $s \sqcap (t_1 \sqcup t_2 \sqcup ...)$ is non-null; so $(s \sqcap t_1) \sqcup (s \sqcap t_2) \sqcup$ ... is non-null; so some $s \sqcap t_i$ is non-null; and so s overlaps some t_i.

A *modalized state space* $\boldsymbol{S}$ - or M-*space*, for short - is an ordered triple $(S, S^\Diamond, \sqsubseteq)$, where $(S, \sqsubseteq)$ is an unmodalized state space and $S^\Diamond$ (possible states) is a non-empty subset of S subject to:

Downward Closure $t \in S^\Diamond$ whenever $s \in S^\Diamond$ and $t \sqsubseteq s$.

We say that a state s is *consistent* or *possible* if $s \in S^\Diamond$ and *inconsistent* or *impossible* otherwise. Note that $\square$ is guaranteed to be a possible state since, given that $S^\Diamond$ contains a member s, its part $\square$ is also a member. A subset T of states is said to be *compatible* if their fusion belongs to $S^\Diamond$ and to be *incompatible* otherwise. Corresponding to each modalized stated space $(S, S^\Diamond, \sqsubseteq)$ is, of course, an unmodalized state space $(S, \sqsubseteq)$.

Of special interest are spaces that contain counterparts of possible worlds. Say that the state s of a modalized space $\boldsymbol{S} = (S, S^\Diamond, \sqsubseteq)$ is a *world-state* if it is consistent and if any consistent state is either a part of s or incompatible with s; and say that the space $\boldsymbol{S}$ itself is a *W-space* if every consistent state of $\boldsymbol{S}$ is part of a world-state. It is often helpful to think of ourselves as working within a W-space, although many of our results will not depend upon this assumption.

A particular kind of W-space may be constructed from the sentential atoms $\mathrm{p}_1, \mathrm{p}_2$, ... of some language. Let us denote the negation $\neg\mathrm{p}$ of an atom p by $\bar{\mathrm{p}}$ and call the atoms p and their negations $\bar{\mathrm{p}}$ *literals*. Then the (*modalized*) *canonical* space $\boldsymbol{S}_c$ *over* the atoms $\{\mathrm{p}_1, p_2, ...\}$ is the triple $(S, S^\Diamond, \sqsubseteq)$, where:

(i) $S = \{L : L$ is a set of literals$\}$;
(ii) $S^\Diamond = \{L \in S$: L does not contain both a sentence letter p and its negation $\bar{\mathrm{p}}\}$; and
(iii) $\sqsubseteq = \{(K, L)$: $K \subseteq L \subseteq S\}$.

In this case, we may write p, for example, in place of {p} or pq in place of {p, q}. It is readily verified that $\boldsymbol{S}_c$ is a W-space whose world-states are all sets of literals containing exactly one of p or p′ for each atom p.

Regular Propositions

Given a state space, we take a (*unilateral*) *proposition* to be a set of states (intuitively, the set of states that verify the proposition). We denote unilateral propositions by the letters ‘P’,‘Q’,‘R’ and the like. Such a proposition $P \subseteq S$ is said to be:

trivial if $\square \in P$
verifiable if P is non-empty
unverifiable or *vacuous* if P is empty
closed (*under fusion*) if $\bigsqcup Q \in P$ for any non-empty subset Q of P
convex if $p \in P$ whenever $q, r \in P$ and $q \sqsubseteq p \sqsubseteq r$,
regular if it is closed and convex, and
semi-regular if it is convex.

We shall use p and the like for an arbitrary member of P and $\boldsymbol{p}$ for $\bigsqcup P$. Given that P is non-empty and closed, $\boldsymbol{p} \in P$ and is the maximal verifier of P. We shall later identify $\boldsymbol{p}$ with the *subject-matter of P*.

There is a certain sense in which regular propositions occupy a span of logical space. Given $R \subseteq S$ and $t \in S$, we take their *span* $[R, t]$ to be $\{s$: for some $r \in R, r \sqsubseteq s \sqsubseteq t\}$. We call R the *lower limit* or *base of* the span and t its *upper limit*. We now have the following simple criteria for when a proposition is regular:

Lemma 1 *The following conditions on the proposition P are equivalent:*

(i) *P is regular*
(ii) *P is convex and either P is empty or* $\boldsymbol{p} \in P$
(iii) $P = [P, \boldsymbol{p}]$
(iv) *P is of the form* $[R, t]$ *for* $R \subseteq S$ *and* $t \in S$.

Proof (i) ⇒ (ii). Suppose P is regular. Then P is convex by definition; and, given that P is non-empty, $\boldsymbol{p} \in P$ by closure.

(ii) ⇒ (iii). Suppose (ii). If P is empty then clearly $P = [P, \boldsymbol{p}]$. If $\boldsymbol{p} \in P$, then $P \subseteq [P, \boldsymbol{p}]$ and $[P, \boldsymbol{p}] \subseteq P$ by P convex.

(iii) ⇒ (iv). Set $R = P$ and $t = \boldsymbol{p}$.

(iv) ⇒ (i). Suppose $P = [R, t]$. Let p be the non-empty fusion $p_1 \sqcup p_2 \sqcup \ldots$ for $p_1, p_2, \ldots \in P$. Then for each i, there is an $r_i \in R$ for which $r_i \sqsubseteq p_i \sqsubseteq t$. Hence $r_i \sqsubseteq p \sqsubseteq t$ and $p \in P$. Now suppose $p \sqsubseteq q \sqsubseteq r$ with p, $r \in P$. Then $p' \sqsubseteq p$ for some $p' \in R$ and $r \sqsubseteq t$. Hence $p' \sqsubseteq q \sqsubseteq t$ and so $q \in P$.

□

Note that a regular proposition P may be identical to many different spans $[R, t]$. The upper limit t of the span must always be $\boldsymbol{p}$ (for P non-empty). The largest lower limit $R \subseteq P$ is P itself, though there will in general be many lower limits and not necessarily any smallest lower limit.

The above lemma provides us with a limited form of monotonicity for regular propositions - given a verifier p, any extension $q \sqsupseteq p$ will also be a verifier as long as it lies within the subject-matter $\boldsymbol{p}$ of the proposition. Thus the current view of propositions provides us with a kind of compromise between insistence on exact relevance without any form of monotonicity and adherence to an unrestricted form of monotonicity.

Given a proposition P, we use P^* for the smallest proposition $Q \supseteq P$ to be closed under fusion, P_* for the smallest proposition $Q \supseteq P$ to be convex, and P^*_* for the smallest regular proposition Q to contain P.

We set:

$T_\square = \{\square\}$
$F_\blacksquare = \{\blacksquare\}$
$F_\square = \emptyset$, and
$T_\blacksquare = S$.

We call $T_\square$, $T_\blacksquare$, $F_\square$ and $F_\blacksquare$ the *extremal* propositions and readily verify that they are regular.

We turn to the bilateral case. A bilateral proposition $\boldsymbol{P}$ is an ordered pair (P, P') where P and P' are unilateral propositions. Intuitively, P is the set of verifiers of the proposition and P' its set of falsifiers. We shall often use P for the first component of $\boldsymbol{P}$, its *positive* content, and P' for the second component, its *negative* content (and similarly when subscripts or other letters are in place). Given $\boldsymbol{P} = (P, P')$, we use $\boldsymbol{p} = \bigsqcup P$ for the positive subject-matter of $\boldsymbol{P}$ and $\boldsymbol{p}' = \bigsqcup P'$ for the negative subject-matter of $\boldsymbol{P}$.

The bilateral proposition $\boldsymbol{P} = (P, P')$ is said to be *convex* (*closed*, *regular*, *semi-regular*) when both its positive content P and its negative content P' are convex (closed, regular, semi-regular); and the bilateral proposition $\boldsymbol{P} = (P, P')$ is said to be *trivial* (*vacuous*) if either P or P' is trivial (vacuous).

We define bilateral analogues of the extremal propositions in the natural way:

$$\begin{aligned}
\boldsymbol{T}_\square &= (T_\square, F_\square)\\
\boldsymbol{T}_\blacksquare &= (T_\blacksquare, F_\blacksquare)\\
\boldsymbol{F}_\square &= (F_\square, T_\square)\\
\boldsymbol{F}_\blacksquare &= (F_\blacksquare, T_\blacksquare).
\end{aligned}$$

Containment and Entailment

Given verifiable propositions P and Q from a state space, we say that Q *contains* P, or that P is a *conjunctive part of* Q, - in symbols, $P \leq_c Q$ - if (i) for all $q \in Q$ there is a $p \in P$ for which $q \sqsupseteq p$ and (ii) for all $p \in P$ there is a $q \in Q$ for which $p \sqsubseteq q$; and we say that P *entails* Q, or that $P$is a *disjunctive part* of Q, - in symbols, $P \leq_d Q$ - if $P \subseteq Q$. In what follows, we shall sometimes think of containment, in a mereological manner, as a relation from *greater* to *lesser* propositions and of entailment, in a logical manner, as a relation from *stronger* to *weaker* propositions and we shall sometimes drop the subscript from $\leq_c$, though never from $\leq_d$. Note that containment is only taken to be defined on verifiable propositions.

We readily verify the mereological behavior of the extremal propositions:

Lemma 2 *For any proposition P,*

(i) $T_\blacksquare \geq_d P$
(ii) $F_\square \leq_d P$
(iii) $F_\blacksquare \not\geq_d T_\square$ *and* $F_\blacksquare \leq_d T_\blacksquare$ *in any non-singleton space* $\boldsymbol{S}$
(iv) $T_\square \leq_c P$ *for* $P \neq F_\square$
(v) $F_\blacksquare \geq_c P$ *for* $P \neq F_\square$.

Proof (i) & (ii) are immediate from the definitions.

(iii) Since $F_\blacksquare$ and $T_\square$ are non-empty and disjoint.

(iv) For each member p of P, $p \sqsupseteq \square$; and if P is non-empty and hence contains a member p, then $\square \sqsubseteq p$.

(v) For each member p of P, $p \sqsubseteq \blacksquare$; and if P is non-empty and hence contains a member p, then $\blacksquare \sqsupseteq p$. □

Clauses (i) and (ii) show that $T_{\blacksquare}$ and $F_{\square}$ are the top and bottom elements with respect to the disjunctive ordering $\leq_{\mathrm{d}}$, clause (iii) shows that $T_{\square}$ and $F_{\blacksquare}$ are incomparable with respect to the disjunctive ordering, and clauses (iv) and (v) show that $T_{\square}$ and $F_{\blacksquare}$ are the bottom and top elements with respect to the conjunctive ordering $\leq_{\mathrm{c}}$. It is important to note that $\leq_{\mathrm{d}}$ is defined on all propositions P while $\leq_{\mathrm{c}}$ is only defined on the propositions $P \neq F_{\square}$ (and so we do not have the usual bilattice structure).

A proposition P is said to be *definite* (or *non-disjunctive*) if it is of the form $P = \{p\}$ for some state p and is otherwise said to be *indefinite* (or *disjunctive*). Definite propositions correspond to world-propositions within the possible worlds semantics and will later be of importance in the account of exclusionary negation. The criterion for containment simplifies when one of the propositions is definite:

Lemma 3 *If both P and Q are verifiable and one is definite, then $P \geq Q$ iff for all $p \in P$ and $q \in Q$, $p \sqsupseteq q$.*

Proof Assume that $P = \{p'\}$ and that Q contains the verifier q' (the other case is similar). Suppose $P \geq Q$. Take any $q \in Q$. Then $p' \sqsupseteq q$ by clause (ii) in the definition of containment and so $p \sqsupseteq q$ for any $p \in P$ and $q \in Q$. Now suppose $p \sqsupseteq q$ for any $p \in P$ and $q \in Q$. Then $p' \sqsupseteq q'$ and $q \sqsubseteq p'$ for each $q \in Q$; and so $P \geq Q$. □

Regular closure normally makes no difference to whether one proposition is contained in another:

Lemma 4 *For verifiable propositions P and Q:*

(i) *$P \geq Q$ implies $P^*_* \geq Q^*_*$, and*
(ii) *$P^*_* \geq Q^*_*$ implies $P \geq Q$ for closed P and Q.*

Proof (i) Suppose $P \geq Q$. Take $p' \in P^*_* = [P, \boldsymbol{p}]$. Then for some $p \in P$, $p \sqsubseteq p' \sqsubseteq \boldsymbol{p}$. Since $P \geq Q$, $p \sqsupseteq q$ for some $q \in Q$ and so $p' \sqsupseteq q \in Q^*_*$. Now take $q' \in Q^*_*$. Then $q' \sqsubseteq \boldsymbol{q} = \bigsqcup Q = \bigsqcup Q^*_*$. But for each $q_{\mathrm{i}} \in Q$ there is a $p_{\mathrm{i}} \in P$ for which $q_{\mathrm{i}} \sqsubseteq p_{\mathrm{i}}$; and so $q' \sqsubseteq \mathbf{q} = q_1 \sqcup q_2 \sqcup \ldots \sqsubseteq p_1 \sqcup p_2 \sqcup \ldots \in P^*_*$.

(ii) Suppose $P^*_* \geq Q^*_*$. Take $p' \in P \subseteq P^*_*$. Then $p' \sqsupseteq q$ for some $q \in Q^*_* = [Q, \boldsymbol{q}]$, where $q' \sqsubseteq q$ for some $q' \in Q$. But then $p' \sqsupseteq q' \in Q$. Now take $q' \in Q \subseteq Q^*_*$. Then $q' \sqsubseteq p$ for some $p \in P^*_* = [P, \boldsymbol{p}]$, where $p \sqsubseteq \boldsymbol{p}$. But then $q' \sqsubseteq p \in P$, given that P is closed.

□

The relation $\geq_{\mathrm{d}}$ is anti-symmetric on all propositions and the relation $\geq_{\mathrm{c}}$ is anti-symmetric on all convex propositions:

Lemma 5 *For arbitrary propositions P and Q, $P \geq_{\mathrm{d}} Q$ and $Q \geq_{\mathrm{d}} P$ implies $P = Q$; and for convex propositions P and Q, $P \geq_{\mathrm{c}} Q$ and $Q \geq_{\mathrm{c}} P$ implies $P = Q$.*

Proof The result for $\geq_d$ is immediate. For the case of $\geq_c$, suppose $P \geq_c Q$ and $Q \geq_c P$ and take $p \in P$. Since $Q \geq_c P$, $p \sqsubseteq q$ for some $q \in Q$. Since $P \geq_c Q$, $p \sqsupseteq q'$ for some $q' \in Q$. Thus $q' \sqsubseteq p \sqsubseteq q$ and so, by Q convex, $p \in Q$. The other direction is proved similarly. □

The result for $\geq_c$ does not hold for arbitrary propositions for, setting $P = \{\text{p}, \text{pqr}\}$ and $Q = \{\text{p}, \text{pq}, \text{pqr}\}$ within the canonical space, $P \geq Q$ and $Q \geq P$. Part of the reason for insisting on convexity is to avoid having distinct propositions which cannot be distinguished in terms of containment.

We should note that, for regular propositions, the second clause in the definition of $P \geq Q$ may be simplified to $\boldsymbol{q} \sqsubseteq \boldsymbol{p}$. For if $\boldsymbol{q} \sqsubseteq p$ for some $p \in P$ then $\boldsymbol{q} \sqsubseteq p \sqsubseteq \boldsymbol{p} \in P$ and if $\boldsymbol{q} \sqsubseteq \boldsymbol{p}$ then, for each $q \in Q$, $q \sqsubseteq \boldsymbol{q} \sqsubseteq \boldsymbol{p} \in P$. Thus the second clause amounts to the subject-matter of Q being part of the subject-matter of P; and so we can see containment as lying within the tradition of analytic implication initiated by Parry [21].

We turn to the bilateral case. Say that the bilateral proposition $\boldsymbol{P} = (P, P')$ is *verifiable* if P is verifiable and *falsifiable* if P' is verifiable. We may then extend the notions of containment and entailment to bilateral propositions $\boldsymbol{P} = (P, P')$ and $\boldsymbol{Q} = (Q, Q')$ as follows:

for verifiable $\boldsymbol{P}$ and $\boldsymbol{Q}$, $\boldsymbol{P} \leq_c \boldsymbol{Q}$ - $\boldsymbol{Q}$ *contains* $\boldsymbol{P}$ - if $P \leq_c Q$ and $P' \leq_d Q'$, and
for falsifiable $\boldsymbol{P}$ and $\boldsymbol{Q}$, $\boldsymbol{P} \leq_d \boldsymbol{Q}$ - $\boldsymbol{P}$ *entails* $\boldsymbol{Q}$ - if $P \leq_d Q$ and $P' \leq_c Q'$.

Note the restriction of $\leq_c$ to verifiable propositions and the restriction of $\leq_d$ to falsifiable propositions. If we wish to talk in the same breath about disjunctive and conjunctive part, then we should require the propositions under consideration to be non-vacuous (a property which will be preserved under negation and non-empty conjunction).

In analogy to lemma 5, we have:

Lemma 6 *For convex bilateral propositions $\boldsymbol{P}$ and $\boldsymbol{Q}$, $\boldsymbol{P} \geq_d \boldsymbol{Q}$ and $\boldsymbol{Q} \geq_d \boldsymbol{P}$ implies $\boldsymbol{P} = \boldsymbol{Q}$ and $\boldsymbol{P} \geq_c \boldsymbol{Q}$ and $\boldsymbol{Q} \geq_c \boldsymbol{P}$ implies $\boldsymbol{P} = \boldsymbol{Q}$.*

We also have the following analogue of lemma 2:

Lemma 7 *For any bilateral proposition $\boldsymbol{P} = (P, P')$,*

(i) $\boldsymbol{T}_\blacksquare \geq_d \boldsymbol{P}$ *for* $P' \neq F_\square$
(ii) $\boldsymbol{F}_\square \leq_d \boldsymbol{P}$ *for* $P' \neq F_\square$
(iii) $\boldsymbol{F}_\blacksquare \not\geq_d \boldsymbol{T}_\square$ *and* $\boldsymbol{F}_\blacksquare \not\geq_d \boldsymbol{T}_\square$ *in any non-singleton space*
(iv) $\boldsymbol{T}_\square \leq_c \boldsymbol{P}$ *for* $P \neq F_\square$
(v) $\boldsymbol{F}_\blacksquare \geq_c \boldsymbol{P}$ *for* $P \neq F_\square$

Proof (i) $\boldsymbol{T}_\square = (T_\square, F_\square)$ and so we need to show that (a) $T_\square \geq_d P$ and that (b) $F_\square \geq_c P$ for $P \neq F_\square$. But (a) follows from lemma 2(i) and (b) from lemma 2(v). The other results are proved similarly. □

Conjunction

Given the verifiable propositions $P_1, P_2, ...$, we let their *conjunction* $P = P_1 \wedge P_2 \wedge ...$ be $\{p_1 \sqcup p_2 \sqcup ... : p_1 \in P_1, p_2 \in P_2, ...\}$. We do not take the conjunction $P_1 \wedge P_2 \wedge ...$ to be defined when one or more of the propositions $P_1, P_2, ...$ is not verifiable; and use of such expressions as '$P_1 \wedge P_2 \wedge ...$' will always presuppose that the context is one in which the conjunction is in fact defined. Note that, when there are no propositions $P_1, P_2, ...$, the resulting conjunction P will be the proposition $T_{\square}$.

Conjunction is indifferent to the order of the conjuncts and also to their repetition in the case of regular propositions:

Lemma 8 (i) *If the multi-sets* $[P_1, P_2, ...]$ *and* $[Q_1, Q_2, ...]$ *of propositions are the same then* $P_1 \wedge P_2 \wedge ... = Q_1 \wedge Q_2 \wedge ...$ *;*
(ii) $P = P \wedge P \wedge ...$ *for any non-empty repeating sequence* $P, P, ...$ *of the closed proposition* P*;*
(iii) *If the sets* $\{P_1, P_2, ...\}$ *and* $\{Q_1, Q_2, ...\}$ *of regular propositions are the same then* $P_1 \wedge P_2 \wedge ... = Q_1 \wedge Q_2 \wedge ...$

Proof (i) Evident from the definition of $P = P_1 \wedge P_2 \wedge ...$ and $Q_1 \wedge Q_2 \wedge ....$
(ii) Suppose $p \in P$. Then $p \in P, p \in P, ...$ and so $p \in P \wedge P \wedge ....$ Now suppose $p \in P \wedge P \wedge ....$ Then for $p_1 \in P, p_2 \in P, ..., p = p_1 \sqcup p_2 \sqcup ...$; and so, by P closed, $p \in P$.
(iii) Any multi-set $[P_1, P_2, ...]$ of propositions can be put in the form $[P_1, P_1, ..., P_2, P_2, ..., ...]$, where the $P_1, P_2, ...$ are pairwise distinct. By (ii), each $P_i \wedge P_i \wedge ... = P_i$ and so $P_1 \wedge P_1 \wedge ... \wedge P_2 \wedge P_2 \wedge ... . = P_1 \wedge P_2 \wedge ....$ Suppose now that the sets $\{P_1, P_1, ..., P_2, P_2, ..., ...\}$ and $\{Q_1, Q_1, ..., Q_2, Q_2, ..., ...\}$ of regular propositions are the same. Then $P_1 \wedge P_1 \wedge ... \wedge P_2 \wedge P_2 \wedge ... . = P_1 \wedge P_2 \wedge ...$, and $Q_1 \wedge Q_1 \wedge ... \wedge Q_2 \wedge Q_2 \wedge ... ... = Q_1 \wedge Q_2 \wedge ...$, and $P_1 \wedge P_2 \wedge ... = Q_1 \wedge Q_2 \wedge ...$ by (i), given that $[P_1, P_2, ...] = [Q_1, Q_2, ...]$; and so $P_1 \wedge P_1 \wedge ... \wedge P_2 \wedge P_2 \wedge ..., ... = Q_1 \wedge Q_1 \wedge ... \wedge Q_2 \wedge Q_2 \wedge ..., ...$ □

The closure of a proposition can be obtained through iterated conjunction:

Lemma 9 *For* P *a verifiable proposition of cardinality* $\mathbf{c}$*:*

(i) $P^* = P \wedge P \wedge ...$ *for* $\mathbf{c}$ *or more* $P, P, ...$*;*
(ii) P *is closed under fusion iff* $P \wedge P \wedge ... = P$ *for any conjunction* $P \wedge P \wedge .....$

Proof (i) Clearly, $P \wedge P \wedge ... \subseteq P^*$. Now suppose $p = p_1 \sqcup p_2 \sqcup ...$ for $p_1, p_2, ... \in P$. Without loss of generality, we can suppose that there are at most $\mathbf{c}$ elements $p_1, p_2, ...$, since repetitions make no difference to the identity of p. But then $p \in P \wedge P \wedge ....$
(ii) Suppose P is closed. Then $P \wedge P \wedge ... = P$ by lemma 8(ii). Now suppose $P \wedge P \wedge ... = P$ for any identical conjunction $P \wedge P \wedge ....$ Take any $p_1, p_2, ... \in P$. Then $p_1 \sqcup p_2 \sqcup ... \in P \wedge P \wedge ... = P$; and so P is closed. □

Containment may be defined in terms of conjunction:

Lemma 10 *For regular verifiable propositions P and Q:*

$$P \geq_c Q \text{ iff } P \wedge Q = P$$
$$\text{iff for some verifiable proposition } R, R \wedge Q = P.$$

Proof Suppose $P \geq_c Q$. Take $p \sqcup q \in P \wedge Q$ for $p \in P$ and $q \in Q$. Since $P \geq_c Q$, $p' \sqsupseteq q$ for some $p' \in P$. Since P is closed, $p \sqcup p' \in P$. So $p \sqsubseteq p \sqcup q \sqsubseteq p \sqcup p'$ with $p, p \sqcup p' \in P$; and, since P is convex, $p \sqcup q \in P$. Now take $p \in P$. Since $P \geq_c Q$, $p \sqsupseteq q$ for some $q \in Q$. But then $p \sqcup q = p \in P \wedge Q$.

Suppose $P \wedge Q = P$. Then evidently, $R \wedge Q = P$ for some verifiable proposition R.

Suppose $R \wedge Q = P$ for some verifiable proposition R. Take $p \in P$. Then $p \in R \wedge Q$; and so p is of the form $r \sqcup q$ for $r \in R$ and $q \in Q$. But then $q \sqsubseteq p$, establishing the first clause of $P \geq_c Q$. Now take $q \in Q$. Pick an $r \in R$. Then $q \sqcup r \in Q \wedge R = P$; and so $q \sqsubseteq q \sqcup r \in P$, establishing the second clause of $P \geq_c Q$. □

It is important to note that the first equivalence above need not hold for arbitrary propositions, since $P \geq_c P$ even though it is not in general true that $P \wedge P = P$. If $P = \{\text{p, q}\}$ within the canonical space, for example, then $P \wedge P = \{\text{p, q, pq}\}$. Also, the second equivalence need not hold if we drop the requirement that the propositions P and Q should be verifiable, since $P \wedge F_{\square} = F_{\square}$ even though it is not in general true that $P \geq_c F_{\square}$.

For the case of arbitrary propositions, we might define a relation $P \succcurlyeq_c Q$ by: $P = Q \wedge R$ for some proposition R. It is then readily shown that $\succcurlyeq_c$ is reflexive and transitive. It may also shown to be antisymmetric. For suppose $P \succcurlyeq_c Q$ and $Q \succcurlyeq_c P$ - that is, $P = Q \wedge R$ for some R and $Q = P \wedge R'$ for some R'. Take $p \in P$. Then for some $q \in Q$ and $r \in R$, $p = q \sqcup r$ and so $q \sqsubseteq p$. But then for some $p' \in P$ and $r' \in R'$, $q = p' \sqcup r'$. So $r' \sqsubseteq p$. But then $p \sqcup r' = p \in P \wedge R' = Q$. The relation $\succcurlyeq_c$ will agree with $\succcurlyeq_c$ on verifiable regular propositions but not on arbitrary propositions.

Regular closure distributes through conjunction:

Lemma 11 *For regular verifiable propositions P_1, P_2, ...:*

$$(P_{1*}^{\ *} \wedge P_{2*}^{\ *} \wedge ...) = (P_1 \wedge P_2 \wedge ...)_*^*.$$

Proof Suppose $s \in (P_1 \wedge P_2 \wedge ...)_*^* = [(P_1 \wedge P_2 \wedge ...), \boldsymbol{p}_1 \sqcup \boldsymbol{p}_2 \sqcup ...]$. So $p_1 \sqcup p_2 \sqcup ... \sqsubseteq s \sqsubseteq \boldsymbol{p}_1 \sqcup \boldsymbol{p}_2 \sqcup ...$ for $p_1 \in P_1$, $p_2 \in P_2$, ... Now $p_\mathrm{i} \sqsubseteq s \sqcap \boldsymbol{p}_\mathrm{i} \sqsubseteq \boldsymbol{p}_\mathrm{i}$ and so $s \sqcap \boldsymbol{p}_\mathrm{i} \in P_\mathrm{i}$. But $s = s \sqcap (\boldsymbol{p}_1 \sqcup \boldsymbol{p}_2 \sqcup ...) = (s \sqcap \boldsymbol{p}_1) \sqcup (\boldsymbol{s} \sqcap \boldsymbol{p}_2) \sqcup ...$ by Distribution and so $s = (s \sqcap \boldsymbol{p}_1) \sqcup (s \sqcap \boldsymbol{p}_2) \sqcup ... \in (P_{1*}^{\ *} \wedge P_{2*}^{\ *} \wedge ...)$.

For the other direction, suppose $s \in (P_{1*}^{\ *} \wedge P_{2*}^{\ *} \wedge ...)$. Then s is of the form $s_1 \sqcup s_2 \sqcup ...$, where, for each i, there is a $p_\mathrm{i} \in P_\mathrm{i}$ for which $p_\mathrm{i} \sqsubseteq s_\mathrm{i} \sqsubseteq \mathrm{p}_\mathrm{i}$. But then $p_1 \sqcup p_2 \sqcup ... \sqsubseteq s_1 \sqcup s_2 \sqcup ... \sqsubseteq \boldsymbol{p}_1 \sqcup \boldsymbol{p}_2 \sqcup ...$ and so $s \in (P_1 \wedge P_2 \wedge ...)_*^*$. □

Note the use of Distribution in establishing $(P_1 \wedge P_2 \wedge ...)^*_* \subseteq (P^*_{1*} \wedge P^*_{2*} \wedge ...)$. Without this assumption, we would need to define the conjunction of the propositions $P_1, P_2, ...$ within a regular domain to be $(P_1 \wedge P_2 \wedge ...)^*_*$ rather than $(P_1 \wedge P_2 \wedge ...)$.

We can now establish the following 'external' or 'algebraic' characterization of conjunction for regular propositions:

Theorem 12 *Suppose that P_1, P_2, ... are regular verifiable propositions and that $P = P_1 \wedge P_2 \wedge ....$ Then:*

(i) $\boldsymbol{p} = \bigsqcup \boldsymbol{p}_i$; *and*
(ii) *P is the least regular verifiable proposition to contain each of P_1, P_2, ...*

Proof (i) $\boldsymbol{p} = \bigsqcup\{p_1 \sqcup p_2 \sqcup ... : p_1 \in P_1, p_2 \in P_2, ...\} = \boldsymbol{p}_1 \sqcup \boldsymbol{p}_2 \sqcup ...$, given that each $\boldsymbol{p}_i$ is the largest member of P_i.

(ii) Suppose that P_1, P_2, ... are regular verifiable propositions. We establish the following facts in turn:

(1) P is verifiable. For each P_i contains a state p_i and so $p_1 \sqcup p_2 \sqcup ... \in P$.

(2) P is regular.

$$\begin{aligned} P &= P_1 \wedge P_2 \wedge ... \\ &= P^*_{1*} \wedge P^*_{2*} \wedge ... \text{since each of } P_1,\ P_2, ... \text{is regular} \\ &= (P_1 \wedge P_2 \wedge ...)^*_* \text{ by the previous lemma} \end{aligned}$$

and so P is regular.

(3) P contains each P_i by lemma 10.

(4) P is the *least* proposition to contain each of P_1, P_2, ... Suppose the regular proposition Q contains each P_i. It follows that $P \leq Q$. For suppose, first, that $p \in P$. Then p is of the form $p_1 \sqcup p_2 \sqcup ...$, with each $p_i \in P_i$. For each i, $p_i \sqsubseteq q_i$ for some $q_i \in Q$ and so $p_1 \sqcup p_2 \sqcup ... \sqsubseteq q_1 \sqcup q_2 \sqcup ... \in Q$. Now suppose $q \in Q$. Then for each i, $q \sqsupseteq p_i$ for some $p_i \in P_i$ and so $q \sqsupseteq p_1 \sqcup p_2 \sqcup ... \in P$.

□

The result that $P = P_1 \wedge P_2 \wedge ...$ is the least proposition to contain each of P_1, P_2, ... does not hold for semi-regular propositions since it is not generally true that $P \wedge P = P$. Indeed, there may not even exist a least upper bound of two propositions P and Q. For set P = {p, r} and Q = {q, s} within the canonical space. Then it may be verified that {pq, rs} and {ps, rq} are both minimal upper bounds of P and Q; and so P and Q cannot have a least upper bound. Thus the standard lattice-theoretic approach to conjunction is not possible within this more general framework and it is not altogether clear what, if anything, should be put in its place. We might perhaps use a containment relation $\geq_C$ that allows any number of propositions to occur on its right, as in $Q \geq_c P_1, P_2, ...$, where the intended meaning of $Q \geq_c P_1, P_2, ...$ is $Q \geq_c P_1 \wedge P_2 \wedge ....$The conjunction $P_1 \wedge P_2 \wedge ...$ would then be the least upper bound with respect to this relation - that is, we would have $P_1 \wedge P_2 \wedge ... \geq_c P_1, P_2, ...$ and $Q \geq_c P_1 \wedge P_2 \wedge ...$ whenever $Q \geq_c P_1, P_2, ...$

Disjunction

We may give different definitions of disjunction, depending upon whether a full, semi-regular or regular domain of propositions is in question. For arbitrary propositions $P_1, P_2, ...,$ their *disjunction* $P = P_1 \vee P_2 \vee ...$ is simply $(P_1 \cup P_2 \cup ...)$; for semi-regular propositions $P_1, P_2, ...,$ their *disjunction* $P = P_1 \vee P_2 \vee ...$ is the convex closure $(P_1 \cup P_2 \cup ...)_*$ of $(P_1 \cup P_2 \cup ...)$; and for regular propositions $P_1, P_2, ...,$ their disjunction $P = (P_1 \vee P_2 \vee ...)$ is the regular closure $(P_1 \cup P_2 \cup ...)^*_*$ of $(P_1 \cup P_2 \cup ...)$. We cannot, in the last two cases, take the disjunction P to be $(P_1 \cup P_2 \cup ...)$ since $(P_1 \cup P_2 \cup ...)$ may be neither convex nor closed.

In analogy to lemma 10, we have the following result (whose proof is straightforward):

Lemma 13 *For any propositions P and Q,*

$$\begin{aligned} P \leq_{\mathrm{d}} Q &\text{ iff } P = Q \vee P. \\ &\text{ iff } \text{ for some proposition } R,\ P = Q \vee R. \end{aligned}$$

We also have the following external characterization of disjunction:

Theorem 14 *Suppose that $P_1, P_2, ...$ are regular propositions and that $P = P_1 \vee P_2 \vee ....$ Then:*

(i) $\boldsymbol{p} = \bigsqcup \boldsymbol{p}_{\mathrm{i}}$*; and*
(ii) *P is the strongest regular proposition to be entailed by each of $P_1, P_2, ...$*

Proof (i) $\boldsymbol{p} = \bigsqcup(P_1 \cup P_2 \cup ...)^*_* = \bigsqcup(P_1 \cup P_2 \cup ...) = \bigsqcup \boldsymbol{p}_{\mathrm{i}}$.
(ii) (1) P is regular by construction. (2) $P_{\mathrm{i}} \subseteq P$. For given $p_{\mathrm{i}} \in P_{\mathrm{i}}$, $p_{\mathrm{i}} \sqsubseteq p_{\mathrm{i}} \sqsubseteq \boldsymbol{p}_{\mathrm{i}} \sqsubseteq \boldsymbol{p}_1 \sqcup \boldsymbol{p}_2 \sqcup ...$ and so $p_{\mathrm{i}} \in P$. (3) If each P_{i} entails Q, for Q regular, then P entails Q. For $(P_1 \cup P_2 \cup ...) \subseteq Q$, given that each P_{i} entails Q; and so $P = (P_1 \cup P_2 \cup ...)^*_* \subseteq Q^*_* = Q$, given that Q is regular. □

A similar result holds for semi-regular propositions and is proved in a similar way. Thus disjunction in the general case, in contrast to conjunction, does have a straightforward lattice-theoretic characterization.

Let $\mathcal{P}$ be the set of all propositions and $\mathcal{P}'$ the set of all verifiable propositions. We may then readily establish the following characterization of the extremal propositions in terms of conjunction and disjunction:

Lemma 15

$$\begin{aligned} T_\square &= \bigwedge \emptyset \\ T_\blacksquare &= \bigvee \mathcal{P} \\ F_\square &= \bigvee \emptyset \\ F_\blacksquare &= \bigwedge \mathcal{P}'. \end{aligned}$$

We have the normal distribution principles for conjunction and disjunction within a regular domain:

Theorem 16 *For regular verifiable propositions P, Q and R:*

(i) $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$
(ii) $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

Proof (i) Suppose $s \in P \wedge (Q \vee R)$. Then s is of the form $p \sqcup t$ for $p \in P$ and $t \in [Q \cup R, \boldsymbol{q} \sqcup \boldsymbol{r}]$. Without loss of generality, assume $q \sqsubseteq t \sqsubseteq \boldsymbol{q} \sqcup \boldsymbol{r}$ for some $q \in Q$. Since $p, q \sqsubseteq s$, $p \sqcup q \sqsubseteq s$ and, since $t \sqsubseteq \boldsymbol{q} \sqcup \boldsymbol{r}$, $t \sqsubseteq \mathrm{p} \sqcup \boldsymbol{q} \sqcup \boldsymbol{r}$. Thus $p \sqcup q \sqsubseteq s \sqsubseteq \boldsymbol{p} \sqcup \boldsymbol{q} \sqcup \boldsymbol{r}$ and, given that $p \sqcup q \in P \wedge Q$, $s \in (P \wedge Q) \vee (P \wedge R)$.

For the other direction, it is clear that $(P \wedge Q) \subseteq P \wedge (Q \vee R)$ and that $(P \wedge R) \subseteq P \wedge (Q \vee R)$; and so $(P \wedge Q) \cup (P \wedge R) \subseteq P \wedge (Q \vee R)$. Hence $(P \wedge Q) \vee (P \wedge R) = [(P \wedge Q) \cup (P \wedge R)]^*_* \subseteq [P \wedge (Q \vee R)]^*_* = [P \wedge (Q \vee R)]$, given that $P \wedge (Q \vee R)$ is regular.

(ii) $P \subseteq (P \vee Q)$ and $P \subseteq (P \vee R)$; and so $P \subseteq (P \vee Q) \wedge (P \vee R)$. Also, $(Q \wedge R) \subseteq (P \vee Q) \wedge (P \vee R)$; and so $P \cup (Q \wedge R) \subseteq (P \vee Q) \wedge (P \vee R)$. But then $P \vee (Q \wedge R) = [P \cup (Q \wedge R)]^*_* \subseteq [(P \vee Q) \wedge (P \vee R)]^*_* = (P \vee Q) \wedge (P \vee R)$, given that $(P \vee Q) \wedge (P \vee R)$ is regular.

For the other direction, take $s \in (P \cup Q) \wedge (P \cup R)$. There are four cases. (a) $s \in P$. But then $s \in P \vee (Q \wedge R)$. (b) s is of the form $p \sqcup r$ for $p \in P$ and $r \in R$. But $p \sqsubseteq p \sqcup r \sqsubseteq \boldsymbol{p} \sqcup \boldsymbol{q} \sqcup \boldsymbol{r}$ and so $s = p \sqcup r \in P \vee (Q \wedge R)$. (c) s is of the form $q \sqcup p$ for $q \in Q$ and $p \in P$. Similar to case (b). (d) s is of the form $q \sqcup r$ for $q \in Q$ and $r \in R$. But then $s \in (Q \wedge R) \subseteq P \vee (Q \wedge R)$. So in each case:

$$(*)(P \cup Q) \wedge (P \cup R) \subseteq P \vee (Q \wedge R).$$

Hence:

$$\begin{aligned}(P \vee Q) \wedge (P \vee R) &= (P \cup Q)^*_* \wedge (P \cup R)^*_* \\ &= [(P \cup Q) \wedge (P \cup R)]^*_* \text{ by lemma 11} \\ &\subseteq [P \vee (Q \wedge R)]^*_* \text{ by } (*) \text{ above} \\ &= P \vee (Q \wedge R)\end{aligned}$$

□

The infinitary form of these results may be proved in the same way. The first distribution principle holds for arbitrary propositions, but the second distribution principle does not even hold for semi-regular propositions. For in the canonical space, let $P = \{p\}$, $Q = \{q\}$ and $R = \{r\}$. Then $P \vee (Q \wedge R) = \{p, qr\}$ while $(P \vee Q) \wedge (P \vee R) = \{p, pr, qp, qr\}$.

Conjunction and Disjunction - the Bilateral Case

Given the bilateral propositions $\boldsymbol{P}_1 = (P_1, P_1')$, $\boldsymbol{P}_2 = (P_2, P_2')$, ..., we let:

$\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge ... = (P_1 \wedge P_2 \wedge ..., P_1' \vee P_2' \vee ...)$ for verifiable $\boldsymbol{P}_1, \boldsymbol{P}_2, ...$; and
$\boldsymbol{P}_1 \vee \boldsymbol{P}_2 \vee ... = (P_1 \vee P_2 \vee ..., P_1' \wedge P_2' \wedge ..)$ for falsifiable $\boldsymbol{P}_1, \boldsymbol{P}_2, ...$

We have some analogues of the previous results for unilateral propositions:

Lemma 17 (i) *For regular verifiable propositions* $\boldsymbol{P} = (P, P')$ *and* $\boldsymbol{Q} = (Q, Q')$:

$$\boldsymbol{P} \geq_c \boldsymbol{Q} \text{ iff } \boldsymbol{P} \wedge \boldsymbol{Q} = \boldsymbol{P} \text{ iff } \boldsymbol{R} \wedge \boldsymbol{Q} = \boldsymbol{P} \text{ for some verifiable proposition } \boldsymbol{R}.$$

(ii) *For regular falsifiable propositions* $\boldsymbol{P} = (P, P')$ *and* $\boldsymbol{Q} = (Q, Q')$:

$$\boldsymbol{P} \geq_d \text{Q iff } \boldsymbol{P} \vee \boldsymbol{Q} = \boldsymbol{P} \text{ iff R } \vee \boldsymbol{Q} = \boldsymbol{P} \text{ for some falsifiable proposition } \boldsymbol{R}$$

Proof (i) Suppose $\boldsymbol{P} \geq_c \boldsymbol{Q}$. Then $P \geq_c Q$ and $P' \geq_d Q'$. By lemmas 10 and 13, $P \wedge Q = P$ and $P' \vee Q' = P'$. But then $\boldsymbol{P} \wedge \boldsymbol{Q} = (P \wedge Q, P' \vee Q') = (P, P') = \boldsymbol{P}$.

Clearly, $\boldsymbol{P} \wedge \mathbf{Q} = P$ implies $\boldsymbol{R} \wedge \boldsymbol{Q} = \boldsymbol{P}$ for some verifiable proposition $\boldsymbol{R}$.

Now suppose $\boldsymbol{R} \wedge \boldsymbol{Q} = \boldsymbol{P}$ for some verifiable proposition $\boldsymbol{R} = (R, R')$. Then $(R \wedge Q, R' \vee Q') = (P, P')$ and so (a) $R \wedge Q = P$ and (b) $R' \vee Q' = P'$. Again by lemmas 10 and 13, $P \geq_c Q$ and $P' \geq_d Q'$. But then $\boldsymbol{P} \geq_c \boldsymbol{Q}$.

(ii) is proved similarly.

□

Theorem 18 *Suppose that* $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ *are regular verifiable propositions and that* $\boldsymbol{P} = \boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge \ldots$. *Then:*

(i) $\boldsymbol{p} = \bigsqcup \boldsymbol{p}_{\mathrm{i}}$ *and* $\boldsymbol{p}' = \bigsqcup \boldsymbol{p}_i'$;
(ii) $\boldsymbol{P}$ *is the least regular proposition to contain each of* $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$.

Proof $\boldsymbol{P}$ is of the form $(P_1 \wedge P_2 \wedge \ldots, P_1' \vee P_2' \vee \ldots)$. From theorem 13(ii), $P_1 \wedge P_2 \wedge \ldots$ is the least proposition to contain each of $P_1, P_2, \ldots$ and, from theorem 14(ii), $P_1' \vee P_2' \vee \ldots$ is the greatest regular proposition to be entailed by each of $P_1', P_2', \ldots$. □

Theorem 19 *Suppose that* $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ *are regular falsifiable propositions and that* $P = \boldsymbol{P}_1 \vee \boldsymbol{P}_2 \vee \ldots$. *Then:*

(i) $\boldsymbol{p} = \bigsqcup \boldsymbol{p}_i$ *and* $\boldsymbol{p}' = \bigsqcup \boldsymbol{p}_i'$;
(ii) $\boldsymbol{P}$ *is the strongest regular proposition to be entailed by each of* $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$.

Proof Similar to the proof of the previous theorem. □

Theorem 20 *For regular non-vacuous propositions* $\boldsymbol{P}, \boldsymbol{Q}$ *and* $\boldsymbol{R}$:

(i) $\boldsymbol{P} \wedge (\boldsymbol{Q} \vee \boldsymbol{R}) = (\boldsymbol{P} \wedge \boldsymbol{Q}) \vee (\boldsymbol{P} \wedge \boldsymbol{R})$
(ii) $\boldsymbol{P} \vee (\boldsymbol{Q} \wedge \boldsymbol{R}) = (\boldsymbol{P} \vee \boldsymbol{Q}) \wedge (\boldsymbol{P} \vee \boldsymbol{R})$.

Proof We may prove (i) by using theorem 16(i) on the 'left' and theorem 16(ii) on the 'right'; and similarly, in reverse, for (ii). □

Note that neither Distribution principle will hold for arbitrary or for semi-regular propositions, since each Distribution principle for bilateral propositions involves both distribution principles for unilateral propositions on the left and on the right.

Let $\mathcal{P}$ be the set of all falsifiable propositions (P, P') and $\mathcal{P}'$ the set of all verifiable propositions (P, P'). Then:

Lemma 21

$$\boldsymbol{T}_{\square} = \bigwedge \emptyset$$
$$\boldsymbol{T}_{\blacksquare} = \bigvee \mathcal{P}$$
$$\boldsymbol{F}_{\square} = \bigvee \emptyset$$
$$\boldsymbol{F}_{\blacksquare} = \bigwedge \mathcal{P}'.$$

These results presuppose that we are working within the full space of bilateral propositions $\boldsymbol{P} = (P, P')$. But they, or analogues of them, also hold under various natural constraints on the domain.

Negation

The treatment of negation is somewhat less straightforward. In the case of unilateral propositions, we may define their negation in terms of a relation of exclusion. Say that a relation $\perp \subseteq S \times S$ is *exclusionary* if it satisfies the following three conditions:

Upward Exclusion if $q \perp p$ and $p \sqsubseteq p^+$ then $q \perp p^+$;
Downward Exclusion if $q \perp p_1 \sqcup p_2 \sqcup \ldots$, then $q \in \{r\colon \text{ for some i, } r \perp p_{\mathrm{i}}\}$;
Null Exclusion never $\square \perp s$ or $s \perp \square$ and, for each $s \neq \square$, there is a t for which $s \perp t$ and a t for which $t \perp s$.

We also say that s *excludes* t when $s \perp t$ holds. Upward Exclusion tells us that an excluder of a state will always exclude an extension of the state; Downward Exclusion tells us that the excluder of the fusion of some states may be derived from the excluders of the component states; and Null Exclusion tells us that a state will exclude or be excluded just in case it is not the null state.

We have formulated Downward Exclusion in a somewhat weak form. In many cases, we can insist upon the following stronger condition:

if $q \perp p_1 \sqcup p_2 \sqcup \ldots$, then q is the fusion of some $q_{\mathrm{k}1}, q_{\mathrm{k}2}, \ldots$, where each q_{kj} excludes some p_{i}.

Note that Downward Exclusion implies part of Null Exclusion, viz., that no state q excludes $\square$, for if q excludes $\square$, which is the empty fusion $p_1 \sqcup p_2 \sqcup \ldots$, then $q \in \{r\colon \text{ for some i, } r \perp p_{\mathrm{i}}\}^*_* = \emptyset^*_* = \emptyset$. For many applications, the second part of Null Exclusion is not required.

Given an exclusionary relation $\perp$, we say that Q *excludes* P if, for each $q \in Q$, there is a $p \in P$ for which $q \perp p$ and, for each $p \in P$, there is a $q \in Q$ for which $q \perp p$ (thus exclusion between propositions is analogous to containment but with the exclusion relation $\perp$ in place of the containment relation $\sqsupseteq$), and we say that the state q *excludes* $P - q \perp P$ - if, for some Q, Q excludes P and $q = \sqcup Q$.

Note that $F_{\blacksquare} = \emptyset$ is the one and only proposition to exclude $F_{\blacksquare}$ and hence $\square = \bigsqcup F_{\square}$ is the one and only state to exclude $F_{\square}$. Also, no state q can exclude a trivial proposition P since q would then have to be a fusion of states one of which excluded $\square$.

We take the *exclusive negation* $-P$ *of* P to be $\{q\colon q \text{ excludes } P\}$ and the *exclusionary negation* $\sim P$ *of* P to be the regular closure $(-P)^*_*$ of $-P$.

Lemma 22 *When* P *is non-trivial then* $\sim P$ *is verifiable and when* P *is verifiable then* $\sim P$ *is non-trivial.*

Proof Suppose $P = \{p_1, p_2, ...\}$ is non-trivial. Since each $p_{\mathrm{i}} \neq \square$, it follows by Null Exclusion that some $q_{\mathrm{i}} \perp p_{\mathrm{i}}$; and so $q = q_1 \sqcup q_2 \sqcup ... \in \sim P$ and $\sim P$ is verifiable.

Now suppose P is verifiable and hence contains a verifier s. Then $\square \notin \sim P$ since, by Null Exclusion, never $\square \perp s$; and hence $\sim P$ is non-trivial.

$\square$

The following two results will later be useful:

Lemma 23 (i) $(-(P^*_*))^*_* = (-P)^*_*$ *for any regular unilateral proposition* P
(ii) $\sim(P_1 \wedge P_2 \wedge ...) = (\sim P_1 \vee \sim P_2 \vee ...)$ *for definite propositions* $P_1, P_2, ...$.

Proof (i) Suppose $q \in -(P^*_*)$ (to show $q \in (-P)^*_*$). Then q is of the form $q_1 \sqcup q_2 \sqcup ...$, where $P^*_* = \{p_1, p_2, ...\}$ and each $q_{\mathrm{i}} \perp p_{\mathrm{i}}$. Now each member of $P = \{p_{\mathrm{k}1}, p_{\mathrm{k}2}, ...\}$ is a member of P^*_* and hence will be excluded by a q_{ki}. Let $\boldsymbol{p}' = q_{\mathrm{k}1} \sqcup q_{\mathrm{k}2} \sqcup ....$ Then $\boldsymbol{p}' \in -P$ and $\boldsymbol{p}' \sqsubseteq q$. Since each q_{i} excludes $p_{\mathrm{i}} \in P^*_*$, with $p_{\mathrm{i}} \sqsubseteq \boldsymbol{p} = p_{\mathrm{k}1} \sqcup p_{\mathrm{k}2} \sqcup ...$, it follows by Upward Exclusion that each $q_{\mathrm{i}} \perp \boldsymbol{p}$; and so, by Downward Exclusion, $q_{\mathrm{i}} \in \{q\colon q \text{ excludes some } p_{\mathrm{kj}}\}^*_*$. Thus each $q_{\mathrm{i}} \sqsubseteq \bigsqcup^*_* -P \in (-P)^*_*$ and so $q = q_1 \sqcup q_2 \sqcup ... \in (-P)^*_*$. Since $-(P^*_*) \subseteq (-P)^*_*$, $(-(P^*_*))^*_* \subseteq (-P)^{*\,*}_{*\,*} = (-P)^*_*$.

Now suppose $q \in -P$ (to show $q \in -(P^*_*)$). Then q is of the form $q_1 \sqcup q_2 \sqcup ...$, where $P = \{p_1, p_2, ...\}$ and each $q_{\mathrm{i}} \perp p_{\mathrm{i}}$. Now for each $p \in P^*_*$, there is a $p_{\mathrm{i}} \in P$ for which $p_{\mathrm{i}} \sqsubseteq p$. Since $q_{\mathrm{i}} \perp p_{\mathrm{i}}$, $q_{\mathrm{i}} \perp p$ by Upward Exclusion. But it is then clear that $q \perp P^*_*$ and so $q \in -(P^*_*)$. Since $-P \subseteq -(P^*_*)$, $(-P)^*_* \subseteq (-(P^*_*))^*_*$.

(ii) Each P_{i} will be of the form $\{p_{\mathrm{i}}\}$ for some state p_{i}. Suppose $q \in \sim(P_1 \wedge P_2 \wedge ...) = \sim\{p_1 \sqcup p_2 \sqcup ...\}$. Then $q \perp p_1 \sqcup p_2 \sqcup ....$ By Downward Exclusion, $q \in \{r\colon r \text{ excludes some } p_{\mathrm{i}}\}^*_*$. But $\{r\colon r \text{ excludes some } p_{\mathrm{i}}\} \subseteq \sim\{p_1\} \cup \sim\{p_2\} \cup ...$; so $\{r\colon r \text{ excludes some } p_{\mathrm{i}}\}^*_* \subseteq [\sim\{p_1\} \cup \sim\{p_2\} \cup ...]^*_*$; and so $q \in \sim\{p_1\} \vee \sim\{p_2\} \vee ...$.

Now suppose $q \in \sim\{p_1\} \vee \sim\{p_2\} \vee ....$ Then for some p_{i} and some $q' \perp p_{\mathrm{i}}$, $q' \sqsubseteq q \sqsubseteq q_1 \sqcup q_2 \sqcup ...$, where each q_{j} excludes some p_{j}. By Upward Exclusion, each q_{j} excludes $p_1 \sqcup p_2 \sqcup ...$; and, since $q' \perp p_{\mathrm{i}}$, it follows, again by Upward Exclusion, that $q' \perp p_1 \sqcup p_2 \sqcup ....$ But then by the convexity of $\sim[\{p_1\} \wedge \{p_2\} \wedge ...]$, $q \in \sim[\{p_1\} \wedge \{p_2\} \wedge ...]$.

$\square$

We can establish the De Morgan Laws for exclusionary negation:

Lemma 24 *For regular propositions $P_1, P_2, ...,$*

(i) $\sim(P_1 \vee P_2 \vee ...) = \sim P_1 \wedge \sim P_2 \wedge ...,\ P_1, P_2, ...$ *non-trivial*
(ii) $\sim(P_1 \wedge P_2 \wedge ...) = \sim P_1 \vee \sim P_2 \vee ...,\ P_1, P_2, ...$ *verifiable;*

Proof (i) Note that if P_1, P_2, ... are non-trivial then $\sim P_1$, $\sim P_2$, ... are verifiable. We then have:

$$\begin{aligned}(\sim P_1 \wedge \sim P_2 \wedge ...) &= (-P_1)^*_* \wedge (-P_2)^*_* \wedge ... \text{ by definition of } \sim \\ &= (-P_1 \wedge -P_2 \wedge ...)^*_* \text{ by lemma 11} \\ &= (-(P_1 \cup P_2 \cup ...))^*_* \text{ by definition of } \wedge \text{ and } - \\ &= (-((P_1 \cup P_2 \cup ...)^*_*))^*_* \text{ by lemma 23} \\ &= (-(P_1 \vee P_2 \vee ...))^*_* \text{ by definition of} \vee \\ &= \sim (P_1 \vee P_2 \vee ...) \text{ by definition of} \sim\end{aligned}$$

(ii) Suppose first that each of P_1, P_2, ... is trivial. Then $P_1 \wedge P_2 \wedge$ ... is trivial; and so $\sim(P_1 \wedge P_2 \wedge ...) = F_\square = F_\square \vee F_\square \vee ... = \sim P_1 \vee \sim P_2 \vee ...$. So we may suppose that one of P_1, P_2, ... is trivial. Now $(P_1 \wedge P_2 \wedge ...) = \bigvee\{\{p_1 \sqcup p_2 \sqcup ...\} : p_1 \in P_1, p_2 \in P_2, ...\}$.
So:
$\sim(P_1 \wedge P_2 \wedge ...) = \sim\bigvee\{\{p_1 \sqcup p_2 \sqcup ...\} : p_1 \in P_1, p_2 \in P_2, ...\}$
$= \bigwedge\{\sim \{p_1 \sqcup p_2 \sqcup ...\} : p_1 \in P_1, p_2 \in P_2, ...\}$ by part (i) given that each $\{p_1 \sqcup p_2 \sqcup ...\}$ is non-trivial

$$\begin{aligned}&= \bigwedge\{\sim\{p_1\} \vee \sim\{p_2\} \vee ... : p_1 \in P_1, p_2 \in P_2, ...\} \text{ by lemma 23(ii)} \\ &= \bigvee_{\mathrm{i}} \bigwedge\{\sim \{p\} : p \in P_{\mathrm{i}}\} \text{ by General Distribution (Theorem 16)} \\ &= \bigvee_{\mathrm{i}} \sim \bigvee\{\{p\} : p \in P_{\mathrm{i}}\} \text{ by (i)} \\ &= \sim P_1 \vee \sim P_2 \vee ....\end{aligned}$$

□

There is no general guarantee that $\sim\sim P = P$. To adapt a previous example, suppose that there are three states r (red), b (blue), g (green) with any two of them excluding the third. Let RED be the proposition $\{r\}$. Then $\sim$RED $= \{b, g, b \sqcup g\}$ while $\sim\sim$RED $= \{r, r \sqcup g, r \sqcup b, r \sqcup g \sqcup b\}$. However, the Law of Double Negation will hold if it holds for all definite propositions:

Theorem 25 *If $\sim\sim P = P$ for each definite unilateral proposition P then $\sim\sim P = P$ for every unilateral proposition whatever.*

Proof This is clear if $P = F_\Box$. So let us suppose P is verifiable. Now $P = \bigvee\{\{p\} : p \in P\}$. So:

$$\begin{aligned}\sim\sim P &= \sim\sim \bigvee\{\{p\} : p \in P\}\\ &= \sim \bigwedge\{\sim\{p\} : p \in P\} \text{ by lemma 24(i)}\\ &= \bigvee\{\sim\sim \{p\} : p \in P\} \text{ by lemma 24(ii)}\\ &= \bigvee\{\{p\} : p \in P\} \text{ by assumption}\\ &= P.\end{aligned}$$

□

Negation - the Bilateral Case

The (classical) negation of a bilateral proposition is simply obtained by reversing its verifiers and its falsifiers. Thus when $\boldsymbol{P} = (P, P')$, $\neg\boldsymbol{P} = (P', P)$.

From the previous definitions, we immediately obtain:

Lemma 26 *For non-vacuous propositions* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$,

(i) $\boldsymbol{P} \leq_{\mathrm{c}} \boldsymbol{Q}$ *iff* $\neg\boldsymbol{P} \leq_{\mathrm{d}} \neg\boldsymbol{Q}$, *and*
(ii) $\boldsymbol{P} \leq_{\mathrm{d}} \boldsymbol{Q}$ *iff* $\neg\boldsymbol{P} \leq_{\mathrm{c}} \neg\boldsymbol{Q}$.

We see from this result that containment and entailment, which seem very different from the unilateral perspective, can be seen to be two sides of the same coin when viewed from the bilateral perspective.

It is readily shown that Double Negation, De Morgan and Distribution hold in the bilateral case:

Theorem 27 (i) $\neg\neg\boldsymbol{P} = \boldsymbol{P}$

(ii) $\neg(\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge ...) = (\neg\boldsymbol{P}_1 \vee \neg\boldsymbol{P}_2 \vee ...)$, $\boldsymbol{P}_1, \boldsymbol{P}_2, ...$ verifiable,
(iii) $\neg(\boldsymbol{P}_1 \vee \boldsymbol{P}_2 \vee ...) = (\neg\boldsymbol{P}_1 \wedge \neg\boldsymbol{P}_2 \wedge ...)$, $\boldsymbol{P}_1, \boldsymbol{P}_2, ...$ falsifiable.

Proof (i) For $\boldsymbol{P} = (P, P')$, $\neg\neg\boldsymbol{P} = \neg\neg(P, P') = \neg(P', P) = (P, P') = \boldsymbol{P}$.
(ii) For $\boldsymbol{P}_i = (P_i, P'_i)$:

$$\begin{aligned}\neg(\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge ...) &= \neg(P_1 \wedge P_2 \wedge ..., P'_1 \vee P'_2 \vee ...)\\ &= (P'_1 \vee P'_2 \vee ..., P_1 \wedge P_2 \wedge ...) \text{ by definition of } \neg\\ &= (P'_1, P_1) \vee (P'_2, P_2) \vee ... \text{ by definition of } \vee\\ &= \neg(P_1, P'_1) \vee \neg(P_2, P'_2) \vee ... \text{ by definition of } \neg\\ &= (\neg\boldsymbol{P}_1 \vee \neg\boldsymbol{P}_2 \vee ...)\end{aligned}$$

(iii) Similarly.

□

Suppose given a set $\mathcal{Q}$ of bilateral propositions. We then let $\mathcal{Q}^{\mathrm{n}}$ be the closure $\mathcal{Q} \cup \{\neg\boldsymbol{P} : \boldsymbol{P} \in \mathcal{Q}\}$ of $\mathcal{Q}$ under negation; and similarly, we let $\mathcal{Q}^{\mathrm{c}}$ be the closure of $\mathcal{Q}$

under conjunction and $\mathcal{Q}^{\mathrm{d}}$ the closure of $\mathcal{Q}$ under disjunction. We also let $\mathcal{Q}^{\mathrm{b}}$ be the closure of $\mathcal{Q}$ under negation, disjunction and conjunction, and let $\mathcal{Q}^{\mathrm{ncd}} = ((\mathcal{Q}^{\mathrm{n}})^{\mathrm{c}})^{\mathrm{d}}$; and similarly for $\mathcal{Q}^{\mathrm{cd}}$ and the like. Note that $\mathcal{Q}^{\mathrm{ncd}}$, in contrast to $\mathcal{Q}^{\mathrm{b}}$, requires that the negations, conjunctions and disjunctions be formed in a certain order. We say, in case $\mathcal{P} = \mathcal{Q}^{\mathrm{b}}$, that $\mathcal{P}$ is the *Boolean closure of* $\mathcal{Q}$; and we say that $\mathcal{P}$ is a *Boolean domain* if $\mathcal{P} = \mathcal{P}^{\mathrm{b}}$, i.e. if $\mathcal{P}$ is closed under the Boolean operations. It is important to bear in mind that, for the purpose of these definitions, conjunction is only defined on verifiable propositions and disjunction only defined on falsifiable propositions.

We have the following abstract version of the standard disjunctive and conjunctive normal form theorems in classical propositional logic:

Theorem 28 *Suppose $\mathcal{P}$ is the Boolean closure of $\mathcal{Q}$. Then:*

(i) $\mathcal{P} = \mathcal{Q}^{\mathrm{ncd}}$, *and*
(ii) $\mathcal{P} = \mathcal{Q}^{\mathrm{ndc}}$

Proof By a straightforward induction on the formation of the propositions in $\mathcal{P}$ using the De Morgan, Distribution and Double Negation Laws. □

We have the following natural requirements on trivial and vacuous propositions:

Triviality Any trivial proposition is vacuous;
Vacuity Any vacuous proposition is trivial;
Strong Triviality Any trivial proposition is identical to $\boldsymbol{T}_{\Box}$ or $\boldsymbol{F}_{\Box}$.

These requirements are preserved under Boolean combination.

Theorem 29 *If the propositional subdomain $\mathcal{Q}$ satisfies Triviality (or Vacuity or Strong Triviality) then so does its Boolean closure $\mathcal{P} = \mathcal{Q}^{\mathrm{b}}$.*

Proof We prove the result by induction on the formation of a proposition $\boldsymbol{P}$ in $\mathcal{Q}^{\mathrm{b}}$. If $\boldsymbol{P}$ is of the form $\neg\boldsymbol{Q}$ then the result is trivial, since it is evident from the symmetry of the requirements that if $\boldsymbol{Q}$ satisfies the requirement then so does $\neg\boldsymbol{Q}$. So, in each case, we need only show that the requirements are preserved under conjunction. We go through each case in turn.

Triviality. Take $\boldsymbol{P}$ of the form $\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge ...$ for verifiable $\boldsymbol{P}_1 = (P_1, P_1')$, $\boldsymbol{P}_2' = (P_2, P_2')$, ... (recall that conjunction is only defined on verifiable propositions), so that $\boldsymbol{P} = (P, P') = (P_1 \wedge P_2 \wedge ..., P_1' \vee P_2' \vee ...)$; and suppose that $\boldsymbol{P}$ is trivial. There are two cases:

(a) P is trivial. But then each P_1, P_2, ... is trivial. By IH, each P_i' is vacuous; so $P' = P_1' \vee P_2' \vee$ ... and hence $\boldsymbol{P}$ is vacuous.
(b) P' is trivial. But then a P_i' is trivial. By IH, P_i is vacuous; and so the conjunction is illegitimate.

Vacuity Take $\boldsymbol{P}$ of the form $\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge ...$ for verifiable $\boldsymbol{P}_1 = (P_1, P_1')$, $\boldsymbol{P}_2 = (P_2, P_2')$, ...; and suppose $\boldsymbol{P}$ is vacuous. There are two cases:

(a) P is vacuous. But then some P_i is vacuous and the conjunction is illegitimate.

(b) P' is vacuous. But then each P_i' is vacuous. By IH, each P_i is trivial; and so $P = P_1 \wedge P_2 \wedge ...$ and hence $\boldsymbol{P}$ is trivial.

Strong Triviality Take $\boldsymbol{P}$ of the form $\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge ...$ for verifiable $\boldsymbol{P}_1 = (P_1, P_1')$, $\boldsymbol{P}_2 = (P_2, P_2')$, ...; and suppose $\boldsymbol{P}$ is trivial. There are two cases:

(a) P is trivial. But then each $P_1, P_2, ...$ is trivial. By IH, each $\boldsymbol{P}_i$ is identical to $\boldsymbol{T}_\square$ or $\boldsymbol{F}_\square$; and hence each $\boldsymbol{P}_i = \boldsymbol{T}_\square$ and $\boldsymbol{P} = \boldsymbol{T}_\square$.

(b) P' is trivial. But then a P_i' is trivial. By IH, $\boldsymbol{P}_i$ is identical to $\boldsymbol{T}_\square$ or $\boldsymbol{F}_\square$ and hence, given that P is trivial, $\boldsymbol{P}_i$ is identical to $\boldsymbol{F}_\square$; and so the conjunction is illegitimate.

$\square$

Given a unilateral proposition P, let $<P>$ be the bilateral proposition $(P, \sim P)$. We might call a proposition of the form $<P>$ *standard*. We have the following results for standard propositions:

Lemma 30 *For regular propositions $P, P_1, P_2, ...$:*

(i) $<P_1 \wedge P_2 \wedge ...> = <P_1> \wedge <P_2> \wedge ...$, *for $P_1, P_2, ...$ verifiable;*
(ii) $<P_1 \vee P_2 \vee ...> = <P_1> \vee <P_2> \vee ...$, *for $P_1, P_2, ...$ non-trivial;*
(iii) $<P> = \bigvee\{<p> : p \in P\}$, *for P non-trivial.*

Proof (i)

$$\begin{aligned} <P_1 \wedge P_2 \wedge ... > &= (P_1 \wedge P_2 \wedge ..., \sim(P_1 \wedge P_2 \wedge ...)) \\ &= (P_1 \wedge P_2 \wedge ..., \sim P_1 \vee \sim P_2 \vee ...) \text{ by lemma 24(ii)} \\ &= (P_1, \sim P_1) \wedge (P_2, \sim P_2) \wedge ... \\ &= < P_1 > \wedge < P_2 > \wedge ... \end{aligned}$$

(ii)

$$\begin{aligned} <P_1 \vee P_2 \vee ... > &= (P_1 \vee P_2 \vee ..., \sim(P_1 \vee P_2 \vee ...)) \\ &= (P_1 \vee P_2 \vee ..., \sim P_1 \wedge \sim P_2 \wedge ...) \text{ by lemma 24(i)} \\ &= (P_1, \sim P_1) \wedge (P_2, \sim P_2) \wedge ... \\ &= <P_1 > \wedge < P_2 > \wedge ... \end{aligned}$$

(iii) For non-trivial $P = \{p_1, p_2, ...\}$,

$$\begin{aligned} <P> &= <\{p_1\} \vee \{p_2\} \vee ... > \\ &= <\{p_1\} \vee < \{p_2\} > \vee ... \text{by part (ii)} \\ &= \bigvee\{< p >: p \in P\}. \end{aligned}$$

$\square$

We turn to the connection between the operations $\sim$ and $\neg$ of exclusionary and classical negation. With any exclusionary relation $\perp$, we may associate two subdomains of propositions:

the *narrow subdomain* $\mathcal{Q}_\perp = \{< \{s\} >: s \in S\}$; and
the *broad subdomain* $\mathcal{Q}^+_\perp = \{< P >: P$ a regular verifiable proposition $\subseteq S\}$

and two corresponding Boolean domains:

the *narrow domain* $\mathcal{P}_\perp = (\mathcal{Q}_\perp)^{\mathrm{b}}$; and
the *broad domain* $\mathcal{P}^+_\perp = (\mathcal{Q}^+_\perp)^{\mathrm{b}}$.

Since each definite proposition $\{s\}$ is regular and verifiable, $\mathcal{Q}_\perp \subseteq \mathcal{Q}^+_\perp$ and hence $\mathcal{P}_\perp \subseteq \mathcal{P}^+_\perp$. For non-trivial P, $< P > = \bigvee\{<p>: p \in P\}$ by lemma 30(iii) and hence $< P > \in \mathcal{P}_\perp$. Thus $\mathcal{P}^+_\perp$ can also be taken to be the Boolean closure $(\mathcal{Q}'_\perp)^{\mathrm{b}}$ of $\mathcal{Q}'_\perp = \mathcal{Q}_\perp \cup \{< P >: P$ trivial $\}$; and so the difference from $\mathcal{P}_\perp$ can be seen to consist in the fact that the propositions in $\mathcal{P}^+_\perp$ can be formed with the help of additional trivial propositions of the form $(P, F_\square)$.

Corollary 31 *The broad domain $\mathcal{P}^+_\perp$ (and hence the narrow domain $\mathcal{P}_\perp$) conform to the Triviality and Vacuity requirements; and the narrow domain $\mathcal{P}_\perp$ also conforms to the Strong Triviality requirement.*

Proof Given theorem 29, we need only show, for the first part, that $\mathcal{Q}^+_\perp$ conforms to the Triviality and Vacuity requirements.

Suppose first that $\boldsymbol{P} = (P, \sim P) \in \mathcal{Q}^+_\perp$ is trivial. There are two cases. (a) P is trivial. But then by lemma 22, $\sim P$ is vacuous and hence $\boldsymbol{P}$ is vacuous. (b) $\sim P$ is trivial. But then by lemma 22, P and hence $\boldsymbol{P}$ is vacuous.

Suppose next that $\boldsymbol{P} = (P, \sim P) \in \boldsymbol{Q}^+_\perp$ is vacuous. There are two cases. (a) P is vacuous. But then by lemma 22, $\sim P$ and hence $\boldsymbol{P}$ is trivial. (b) $\sim P$ is vacuous. But then by lemma 22, P and hence $\boldsymbol{P}$ is trivial.

We now show that $\boldsymbol{Q}_\perp$ conforms to Strong Triviality. Suppose that $\boldsymbol{P} = (P, \sim P) \in \boldsymbol{Q}_\perp$ is trivial. There are two cases. (a) P is trivial. But then P must be of the form $\{\square\}$, $\sim P = F_\square$ and so $\boldsymbol{P} = \boldsymbol{T}_\square$. (b) $\sim P$ is trivial. But by Null Exclusion, this is impossible given that P is definite. $\square$

Corollary 32 *For any exclusionary relation $\perp$, $\mathbf{F}_\square, \mathbf{T}_\square \in \mathcal{P}_\perp$ and hence $\mathbf{F}_\square, \mathbf{T}_\square \in \mathcal{P}^+_\perp$ but $\mathbf{F}_\blacksquare, \mathbf{T}_\blacksquare \notin \mathcal{P}^+_\perp$ and hence $\mathbf{F}_\blacksquare, \mathbf{T}_\blacksquare \notin \mathcal{P}_\perp$.*

Proof $\boldsymbol{T}_\square = (T_\square, F_\square) = (\{\square\}, \sim\{\square\}) \in \mathcal{Q}_\perp \subseteq \mathcal{P}_\perp$; and so $\mathbf{F}_\square = \neg\mathbf{T}_\square \in \mathcal{P}_\perp$. For the second part, we need only note that $\mathbf{F}_\blacksquare$ and $\mathbf{T}_\blacksquare$ are trivial yet non-vacuous propositions. $\square$

Although $\mathcal{P}_\perp$ and $\mathcal{P}^+_\perp$ do not contain $\mathbf{F}_\blacksquare$ or $\mathbf{T}_\blacksquare$, they contain counterparts of $\boldsymbol{F}_\blacksquare$ and $\mathbf{T}_\blacksquare$. For let $T'_\blacksquare = T_\blacksquare - \{\square\}$, $\mathbf{F}'_\blacksquare = (F_\square, T'_\blacksquare)$ and $\mathbf{T}'_\blacksquare = (T_\blacksquare, F'_\blacksquare)$. Then within $\mathcal{P}_\perp$ and $\mathcal{P}^+_\perp$, $\mathbf{F}'_\blacksquare$ will be the conjunction of all verifiable propositions and $\mathbf{T}'_\blacksquare$ the disjunction of all falsifiable propositions.

The next result relates exclusion to falsification and provides conditions under which all falsifiers will be excluders or all excluders will be falsifiers within a narrow domain:

Theorem 33 *Suppose that $P' \subseteq \sim\sim P$ (resp. $P \supseteq \sim\sim P$) holds for all the definite propositions in $\mathcal{Q}_\perp$. Then $P' \subseteq \sim P$ (resp. $P' \supseteq \sim P$) holds for all propositions $\boldsymbol{P} = (P, P')$ in $\mathcal{P}_\perp$.*

Proof $\mathcal{P}$ is of the form $\mathcal{Q}^{\text{ncd}}$ for $\mathcal{Q}$ a definite domain $\mathcal{Q}_\perp$; and so we may prove the results by induction on the generation of $\boldsymbol{P}$ via $\mathcal{Q}^{\text{ncd}}$. We focus on the case $P' \subseteq \sim P$, since the proof for $P' \supseteq \sim P$ is exactly the same but with $\supseteq$ replacing $\subseteq$.

Suppose $\boldsymbol{P} = (P, P') \in \mathcal{Q}$. The result is immediate since $P' = \sim P$.

Suppose $\boldsymbol{P}$ is of the form $\neg\boldsymbol{Q}$ for $\boldsymbol{Q} = (\{p\}, Q') \in \mathcal{Q}$. Thus $\boldsymbol{P} = (Q', \{p\})$, where $P = Q'$, $P' = \{p\}$ and $Q' = \sim P'$. By the supposition, $P' \subseteq \sim\sim P' = \sim Q' = \sim P$.

Suppose $\boldsymbol{P}$ is of the form $\boldsymbol{Q} \wedge \boldsymbol{R}$ for verifiable $\boldsymbol{Q} = (Q, Q')$, $\boldsymbol{R} = (R, R')$ and $\boldsymbol{P} = (P, P') = (Q \wedge R, Q' \vee R')$ (the general case is similar). By IH, $Q' \subseteq \sim Q$ and $R' \subseteq \sim R$; and so $P' = Q' \vee R' \subseteq \sim Q \vee \sim R = \sim(Q \wedge R) = \sim P$, by De Morgan for $\sim$.

Suppose $\boldsymbol{P}$ is of the form $\boldsymbol{Q} \vee \boldsymbol{R}$. Similar to the previous case. □

Corollary 34 *Suppose that $P = \sim\sim P$ holds for all definite propositions P. Then, for any proposition $\boldsymbol{P} = (P, P')$ in $\mathcal{P}_\perp$, $P' = \sim P$ and hence each proposition $\boldsymbol{P} = (P, P')$ in $\mathcal{P}_\perp$ is of the form $<P>$.*

The assumption that $P \subseteq \sim\sim P$ is quite reasonable and hence it is quite reasonable to suppose that any falsifier of a bilateral proposition $\boldsymbol{P} = (P, P')$, i.e. any member of P', is also an excluder, i.e. a member of $\sim P$. This means that, given a narrow domain of propositions $\mathcal{P}_\perp$, we can take the (maximal) underlying exclusionary relation $\perp$ from which it is generated to be $\{(s, t)$: for some proposition $\boldsymbol{P} = (\{s\}, P')$, $t \in P'\}$, thereby determining $\perp$ from $\mathcal{P}_\perp$.

The converse assumption that $P \supseteq \sim\sim P$, by contrast, is not at all reasonable, as is evident from the red/green/blue example above; and so we cannot, in general, assume that the falsifiers of a given bilateral proposition will be a function of its verifiers.

Classical Truth-conditions

We turn to the modal aspects of the theory of Boolean propositions. We should now take ourselves to be working within a modalized state space. Within such a space, there are two important conditions that may be imposed on the relationship between positive and negative content. Say that a bilateral proposition $\boldsymbol{P} = (P, P')$ is *exclusive* if no member of P is compatible with a member of P', that $\boldsymbol{P}$ is *exhaustive* if any consistent state is compatible with a member of P or with a member of P', and that $\boldsymbol{P}$ is *classical* if it is both exclusive and exhaustive.

Within a W-space, these conditions take a more familiar form. Given an arbitrary bilateral proposition $\boldsymbol{P} = (P, P')$ and state s, say $s || \!> \boldsymbol{P}$ if $s \sqsupseteq p$ for some member p of P and that $s <\!|| \boldsymbol{P}$ if $s \sqsupseteq p'$ for some member p' of P'. In the particular case in which s is a world- state w, we say $\boldsymbol{P}$ is *true* (*false*) *at* w if $w || \!> \boldsymbol{P}$ ($w <\!|| \boldsymbol{P}$).

Theorem 35 *Within a W-space:*

(i) ***P*** *is exclusive iff there is no world-state at which* ***P*** *is glutty, i.e. both true and false;*
(ii) ***P*** *is exhaustive iff there is no world-state at which P is gappy, i.e. neither true nor false:*
(iii) ***P*** *is classical iff at any world-state* ***P*** *is bivalent, i.e. either true or false but not both.*

Proof (i) Suppose $\boldsymbol{P}$ is not exclusive. Then some state $p \in P$ is compatible with a state $p' \in P'$. So $p \sqcup p'$ is consistent and hence is contained in a world-state w. But then w contains both p and p' and $\boldsymbol{P}$ is both true and false at w. Now suppose $\boldsymbol{P}$ is both true and false at the world-state w, so that P contains both a $p \in P$ and a $p' \in P'$. Then since p and p' are both parts of w, they are compatible, contrary to the exclusivity of $\boldsymbol{P}$.
(ii) Suppose $\boldsymbol{P}$ is exhaustive and take a world-state w. Since w is consistent, it is compatible with a $p \in P$ or with a $p' \in P'$. Hence w contains such a p or a p' and $\boldsymbol{P}$ is either true or false. Now suppose $\boldsymbol{P}$ is either true or false at each world-state, so that each world-state contains a $p \in P$ or a $p' \in P'$. Take any consistent state q. Then q is part of a world-state w; and since w is compatible with a p or a p', then so is q.
(iii) From (i) and (ii). □

Theorem 36 *For arbitrary propositions* $\boldsymbol{P} = (P, P')$, $\boldsymbol{P}_1 = (P_1, P_1')$, $\boldsymbol{P}_2 = (P_2, P_2')$, ... *and any state s:*

(i)(a) $s||\!> \neg\boldsymbol{P}$ iff $s <\!|| \boldsymbol{P}$
(b) $s <\!|| \neg\boldsymbol{P}$ iff $s||\!>\boldsymbol{P}$
(ii)(a) $s||\!>(\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge ...)$ iff $s||\!>\boldsymbol{P}_1$ and $s||\!>\boldsymbol{P}_2$ *and* ...
(b) $s<\!|| (\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge ...)$ iff $s<\!||\boldsymbol{P}_1$ or $s<\!||\boldsymbol{P}_2$ *and* ...
(iii)(a) $s||\!>(\boldsymbol{P}_1 \vee \boldsymbol{P}_2 \vee ...)$ iff $s||\!>\boldsymbol{P}_1$ or $s||\!>\boldsymbol{P}_2$ *or* ....
(b) $s<\!|| (\boldsymbol{P}_1 \vee P_2 \vee ...)$ iff $s <\!|| P_1$ and $s<\!||\boldsymbol{P}_2$ and ....

Proof (i)(a) For $\boldsymbol{P} = (P, P')$, $s||\!> \neg\boldsymbol{P}$ iff $s \sqsupseteq$ a member of P' given that $\neg\boldsymbol{P} = (P', P)$, and this holds iff $s <\!|| \boldsymbol{P}$.
(i)(b) Similar.
(ii)(a) $s||\!> (\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge ...)$ iff $s \sqsupseteq$ a member of $(P_1 \wedge P_2 \wedge ...)$

iff $s \sqsupseteq p_1 \sqcup p_2 \sqcup$ for some $p_1 \in P_1$, $p_2 \in P_2$, ...
iff $s||\!> \boldsymbol{P}_1$ and $s||\!> \boldsymbol{P}_2$ and ....

(ii)(b) $s <\!|| (\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge ...)$ iff $s \sqsupseteq$ a member of $(P_1' \vee P_2' \vee ...)$

iff $s \sqsupseteq$ a member of $P_1' \cup P_2' \cup ...$
iff $s <\!||\boldsymbol{P}_1$ or $s <\!||\boldsymbol{P}_2$ and ...

(iii)(a)-(b) Similar to (ii) or from (i) and (ii) by De Morgan. □

Combining the previous two results, we can derive the classical truth-conditions for Boolean compounds of bivalent propositions:

Corollary 37 *For bivalent propositions $\boldsymbol{P}, \boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ and any world-state w:*

(i) $w| = \neg\boldsymbol{P}$ iff not $w| = \boldsymbol{P}$
(ii) $w| = (\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge \ldots)$ iff $w| = \boldsymbol{P}_1$ and $w| = \boldsymbol{P}_2$ and ...
(iii) $w| = (\boldsymbol{P}_1 \vee \boldsymbol{P}_2 \vee \ldots)$ iff $w| = \boldsymbol{P}_1$ or $w| = \boldsymbol{P}_2$ or ....

We can also show that the properties of being exclusive or being exhaustive are inherited under Boolean compounding:

Theorem 38 (i)(a) *If $\boldsymbol{P}$ is exclusive then so is $\neg\boldsymbol{P}$*
(b) *If $\boldsymbol{P}$ is exhaustive then so is $\neg\boldsymbol{P}$*
(ii)(a) *If $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, ... are exclusive then so is $(\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge \ldots)$*
(b) *If $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, ... are exhaustive then so is $(\boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge \ldots)$ - within a W-space*
(iii)(a) *If $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, ... are exclusive then so is $(\boldsymbol{P}_1 \vee \boldsymbol{P}_2 \vee \ldots)$*
(b) *If $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, ... are exhaustive then so is $(\boldsymbol{P}_1 \vee \boldsymbol{P}_2 \vee \ldots)$ - within a W-space.*

Proof (i)(a)-(b). By the symmetry of the properties of being exclusive and being exhaustive.

(ii) Take $\boldsymbol{P}_1 = (P_1, P_1')$, $\boldsymbol{P}_2 = (P_2, P_2')$ ..., so that $\boldsymbol{P} = (P, P') = ((P_1 \wedge P_2 \wedge \ldots), (P_1' \vee P_2' \vee \ldots))$. To establish (a), suppose that $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, ... are exclusive and suppose that some member p of $(P_1 \wedge P_2 \wedge \ldots)$ is compatible with a member p' of $(P_1' \vee P_2' \vee \ldots)$. Then $p'' \sqsubseteq p'$ for some $p'' \in P$ and, given that p is of the form $p_1 \sqcup p_2 \sqcup \ldots$ for $p_1 \in P_1$, $p_2 \in P_2$, ..., $p_i \in P_i$ is compatible with $p'' \in P_i'$.

To establish (b), suppose that $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, ... are exhaustive and take an arbitrary possible state s. Then some world-state $w \sqsupseteq s$. There are two cases. (1) w is compatible with a member of each P_i and hence $w \sqsupseteq p_i \in P_i$ for each $i = 1, 2, \ldots$. But then $w \sqsupseteq p = p_1 \sqcup p_2 \sqcup \ldots \in P$and so s is compatible with p. (2) w is not compatible with any member of some P_i. But then w and hence s is compatible with a member of P_i'.

(iii) Proved similarly to (ii) or on the basis of (ii) by De Morgan.

□

The results under (ii)(b) and (iii)(b) can be proved within an arbitrary state space for finitary conjunctions and disjunctions but not for infinitary conjunctions or disjunctions (see Fine [10] for further consideration of this case).

Say that a domain $\boldsymbol{Q}$ of bilateral propositions is *bivalent* (*classical*) if each proposition in $\boldsymbol{Q}$ is bivalent (*classical*). From the theorem, we immediately obtain:

Corollary 39 *If $\boldsymbol{Q}$ is bivalent (within a W-space) then so is $\boldsymbol{Q}^{\mathrm{b}}$.*

We consider the special case in which $\boldsymbol{Q}$ is a basis for the broad domain $\mathcal{P}^{+}_{\perp}$. Say that the exclusionary relation $\perp$ is *classical* if:

(i) if $s \perp t$ then s is incompatible with t;
(ii) if s and t are consistent and s is incompatible with t then, for some $s' \sqsubseteq s$, $s' \perp t$; and
(iii) if t is inconsistent then every consistent state is compatible with some t' for which $t' \perp t$.

Corollary 40 *If $\perp$ is classical then so is $\mathcal{P}^{+}_{\perp}$.*

Proof By the previous corollary, it suffices to show that $\mathcal{Q}'_{\perp} = \mathcal{Q}_{\perp} \cup \{< P > : P$ trivial$\}$ is classical, given that $\perp$ is classical. It is evident that trivial $< P >$ satisfies exclusivity, since either P or $\sim P$ is empty, and it is evident that it satisfies Exhaustivity, since either P or $\sim P$ contains $\Box$, which is compatible with any state. So it remains to show that $\mathcal{Q}_{\perp}$ is classical. Take a proposition ($\{p\}$, $\sim$ $\{p\}$) in $\mathcal{Q}_{\perp}$ and some arbitrary member q of $\sim\{p\}$. Then $q \sqsupseteq q'$ for some q' which excludes p. But then, by condition (i) above, q' and hence q is incompatible with p. For Exhaustivity, suppose s is consistent yet incompatible with p. If p is inconsistent, then it follows from (iii) that $s$is compatible with a $p' \perp p$. So suppose p is consistent. Then by (ii), for some $s' \sqsubseteq s$, $s' \perp p$. But then s is compatible with $s' \in \sim\{p\}$. $\Box$

Semantics and Logic

Our focus has been on developing an abstract theory of truthmaker content, but it is worth pointing out how the abstract theory might relate to the interpretation of a formal language (for further discussion, consult Fine [7]). We suppose given an infinitude of sentence letters p_1, p_2, ... Formulas are then formed from these sentence letters using the Boolean operators $\wedge$, $\vee$, and $\neg$ in the usual way (the double use of the constants $\wedge$, $\vee$, and $\neg$ as operators and as operations should cause no confusion). A (*state*) *model* $\boldsymbol{M}$ is an ordered triple (S, $\sqsubseteq$, $|\cdot|$), where (S, $\sqsubseteq$) is a state space and $|\cdot|$ (valuation) is a function taking each sentence letter p into a bilateral proposition, i.e. a pair (V, F) of subsets of S. When $|\mathrm{p}| = (V, F)$, we let $|\mathrm{p}|^{+} = V$ and $|\mathrm{p}|^{-} = F$.

Given a model $\boldsymbol{M} = (S, \sqsubseteq, |\cdot|)$, we define what it is for an arbitrary formula A to be *verified by* a state s (s||- A) or *falsified by* the state s (s -|| A):

(i)$^{+}$ s ||- p if $s \in |\mathrm{p}|^{+}$;
(i)$^{-}$ s -|| p if $s \in |\mathrm{p}|^{-}$;
(ii)$^{+}$ s ||- ¬B if s -|| B;
(ii)$^{-}$ s -|| ¬B if s ||- B;
(iii)$^{+}$ s ||- B $\wedge$ C if for some t and u, t ||- B, u ||- C and $s = t \sqcup u$;
(iii)$^{-}$ s -|| B $\wedge$ C if s -|| B or s -|| C;
(iv)$^{+}$ s ||- B $\vee$ C if s ||- B or s||- C;
(iv)$^{-}$ s -|| B $\vee$ C if for some t and u, t -|| B, u -|| C and $s = t \sqcup u$.

Alternatively, we can think in terms of extending the map |p| from sentence letters to bilateral propositions to all formulas whatever. Thus in place of clauses (ii)-(iv)(a)(b), we have:

(ii) $|\neg B| = \neg|B|$
(iii) $|B \wedge C| = |B| \wedge |C|$
(iv) $|B \vee C| = |B| \vee |C|$,

using the previous definitions of propositional negation, conjunction and disjunction.

Let us temporarily use [A] ambiguously for |A|, $|A|^*$, $|A|_*$ and $|A|^*_*$, thereby reflecting the various closure conditions that might be imposed upon a proposition. We may then distinguish the following notions of verification (relative to a given model $\boldsymbol{M}$):

Exact Verification $s \Vdash$ A if $s \in$ [A]
Inexact Verification $s | \trianglerighteq$ A if for some $s' \sqsubseteq s$, $s' \Vdash$ A
Loose Verification $s \models$ A if any state compatible with s is compatible with some $t \in$ [A].

These notions of verification will be ambiguous given the ambiguous meaning of [A], but it turns out that the ambiguity only matters in the case of exact verification; that is, inexact and loose verification will coincide whatever the underlying notion of verification given by [A]. Note that the classical notion of loose verification must be defined by reference to a *modalized* state space or model, in contrast to the notions of exact and inexact verification; and, indeed, it is a general characteristic feature of the classical approach that it takes heed of the distinction between consistent and inconsistent states.

Given these different notions of verification, we may distinguish corresponding notions of consequence (either relative to a given model $\boldsymbol{M}$ or allowing $\boldsymbol{M}$ to vary arbitrarily):

Exact Consequence A $\models_e$ C if $s \Vdash$ A implies $s ||$ C
Inexact Consequence A $\models_i$ C if $s | \trianglerighteq$ A implies $s || >$ C
Analytic Consequence (Containment) A $>$ C if [A] $\geq_c$ [C]
Loose (or Classical) Consequence A $\models_l$ C if $s \models$ A implies $s \models$ C.

There will, of course, be corresponding notions of equivalence, according to which A and C are *exactly* (*inexactly*, *analytically*, *classically*) equivalent if each is exactly (inexactly, analytically, loosely) a consequence of the other.

The logic of each of these notions of consequence (or equivalence) can be determined. This is not my task here, but we may note that loose consequence corresponds to the familiar notion of classical consequence, containment for regular non-vacuous propositions [A] and [C] to Angell's logic of analytic entailment [1], inexact consequence to the logic of first degree entailment, and exact consequence to a logic that has been explored in Correia [5] and in unpublished work of the author and Mark Jago.

Minimal Verifiers

Our own account of partial content is hyper-intensional; intensionally equivalent - even truth-functionally equivalent - propositions may not contain or be contained in the same propositions. Yablo and Gemes have developed intensional accounts of partial content, in which intensionally equivalent propositions will be indistinguishable with respect to their containment relationships. In the present section, we discuss Yablo's approach in terms of 'minimal' verifiers; and, in the next section, we discuss Gemes' approach in terms of what I shall call 'compact' verifiers. In each case, we shall show how to accommodate the approach within the more abstract setting of our own framework and show how it may be extended, though with some artificiality, to cases in which the required minimal or compact verifiers do not exist.

Recall that $s \models P$ is loose verification (we must therefore be working within a modalized space). We say that *s-minimally verifies* the proposition P - in symbols, $s|=^{\mathrm{m}} P$ - if $s|= P$ and for no $s' \sqsubset s$ does $s'| = P$. The proposition P itself is said to be *minimal* if (i) $p|=^{\mathrm{m}} P$ for any $p \in P$ and (ii) for any $p|= P$ there is a $p' \in P$ for which $p' \sqsubseteq p$. It follows that, when P is minimal, then every minimal verifier p' of P will belong to P, since there will be a $p \in P$ for which $p \sqsubseteq p'$ and hence for which $p = p'$. Thus a minimal proposition will be identical to its set of minimal verifiers.

Let us say that two propositions P and Q are *loosely* (or *intensionally*) *equivalent* if their loose verifiers are the same (this is equivalent, within a W-space, to their being true in the same possible worlds). Then it is readily shown that any two intensionally equivalent minimal propositions P and Q will be identical. For suppose $p \in P$. Then it will loosely verify P and so will loosely verify Q. If p is a minimal verifier of Q, then $p \in Q$ and we are done. If p is not a minimal verifier of Q, then $p' \in Q$ for some $p' \sqsubset p$. But then p' is a loose verifier of Q and hence of P and so, for some $p'' \in P$, $p'' \sqsubseteq p' \sqsubset p$, and p is not a minimal verifier of P after all.

Partial content (and some other notions) can be defined for minimal propositions in the usual way. Thus given minimal propositions P and Q, we can say that $P \geq Q$ if every verifier of P contains a verifier of Q and every verifier of Q is contained in a verifier of P. The definition may then be extended to propositions P' and Q' intensionally equivalent to minimal propositions P and Q by taking $P' \geq Q'$ to hold just in case $P \geq Q$ holds. However, it is not generally true that a proposition P is intensionally equivalent to a minimal proposition P'; and so one must find some other way of extending the definition to arbitrary propositions.

One way to do this is in terms of 'approximations' to minimal propositions. A proposition Q is said to be *minimalist* if (i) $q \in Q$, $q' \sqsubseteq q$ and $q'|= Q$ implies $q' \in Q$ and (ii) for any $q|= Q, q \sqsupseteq q'$ for some $q' \in Q$. (i) is a downward closure condition and (ii) is an adequacy condition to the effect that every loose verifier must lie above a verifier in the proposition. Q is said to be a *minimalist basis for P* if Q is a minimalist proposition intensionally equivalent to P. Note that a minimalist basis Q for a proposition P will always exist since we can let Q be the set of all loose verifiers of P; and note also that any minimal proposition P will be a minimalist basis for any intensionally equivalent proposition.

We may now extend the relation of partial content to all propositions as follows (Y stands for Yablo):

$P \geq^{\mathrm{Y}} Q$ if for any minimalist basis P' for P there is a minimalist basis Q' for Q for which $P' \geq Q'$.

Thus no matter how closely we approximate to P via a minimalist basis we can approximate sufficiently closely to Q to ensure that the one proposition contains the other.

We would like to show that this definition agrees with the standard definition of partial content in its application to minimal propositions. We first show:

Lemma 33 *Suppose that P is a minimal proposition and that P' is a minimalist basis for P. Then $P' \geq P$.*

Proof (a) Suppose first that $p' \in P'$. Since P' is intensionally equivalent to P, $p' \models P$ and, since P is minimal, $p' \sqsupseteq p$ for some $p \in P$. (b) Now suppose $p \in P$. Since P' is intensionally equivalent to P, $p \models P'$ and, since P' is a minimalist basis for P, $p \sqsupseteq p'$ for some $p' \in P'$. If $p = p'$, we are done. So suppose $p \neq p'$. Then $p' \models P$ and p is not a minimal verifier of P (in fact, this shows $P \subseteq P'$). □

We now establish:

Theorem 34 *For minimal propositions P and Q, $P \geq Q$ iff $P \geq^{\mathrm{Y}} Q$.*

Proof Suppose $P \geq Q$ for minimal propositions P and Q. Take any minimalist basis P' for P. Then $P' \geq P$ by the lemma. We let the minimalist basis Q' for Q be Q itself. Given that $P \geq Q$, $P' \geq P \geq Q'$; and so $P \geq^{\mathrm{Y}} Q$. Now suppose $P \geq^{\mathrm{Y}} Q$. Given that P is minimal, it is a minimalist basis for P. So for some minimalist basis Q' for Q, $P \geq Q'$. But $Q' \geq Q$ by the lemma; and so $P \geq Q$. □

Compact Verifiers

Gemes develops his account of partial content in terms of what one might call compact verification. Given a sentential formula A, a sentence letter will be relevant to A if 'toggling' its truth-value within some assignment of truth-values (i.e. changing its true-value from T to F) can make a difference to the truth-value of A. A compact verifier A is then an assignment of truth-values to the relevant sentence-letters of A under which it is true; and given the notion of a compact verifier, partial content can be defined in the usual way.

For our purposes, such a definition of partial content is far too linguistic; we need a definition that will apply purely at the level of content, without any reference to language. There are perhaps various ways to do this, but one fairly natural way is based on the idea that the assignments of truth-values to the relevant sentence letters will provide a minimally sufficient basis for settling the truth-value of the given formula.

To translate this idea to a more abstract setting, we make the following definitions (which are also of some independent interest). Given some subject-matter $\boldsymbol{s}$, we say that w is an $\boldsymbol{s}$*-world* if w is a consistent part of $\boldsymbol{s}$ and any part of $\boldsymbol{s}$ is either a part of w or inconsistent with w. An $\boldsymbol{s}$-world is a kind of mini-world with respect to the subject-matter $\boldsymbol{s}$. When $\boldsymbol{s}$ is the full state ■, then an $\boldsymbol{s}$-world is a full world, or what we previously called a world-state.

The subject-matter $\boldsymbol{s}$ is said to be *comprehensive* if any consistent state is compatible with some $\boldsymbol{s}$-world. In case $\boldsymbol{s}$ is the full state ■, it will be comprehensive just in case the space is what we called a W-space, in which every consistent state is part of a world-state. The subject-matter $\boldsymbol{s}$ is said to *comprehend* the proposition P if it is comprehensive and every $\boldsymbol{s}$-world loosely verifies or loosely falsifies P; and $\boldsymbol{s}$ is said to be *compact for* P if it comprehends P and if $\boldsymbol{s} \sqsubseteq \boldsymbol{t}$ for any subject-matter $\boldsymbol{t}$ that comprehends P. We now say that *s compactly verifies* the proposition P - in symbols, $s| =^{\mathrm{c}} P$- if $s| = P$ and s is an $\boldsymbol{s}$-world for some compact subject-matter $\boldsymbol{s}$ for P.

Partial content can now be defined for compact propositions in the usual way. Thus given two compact propositions P and Q, we may say that $P \geq Q$ if every compact verifier of P contains a compact verifier of Q and every compact verifier of Q is contained in a compact verifier of P; and the definition may be extended to propositions P' and Q' intensionally equivalent to the compact propositions P and Q by taking $P' \geq Q'$ to hold just in case $P \geq Q$ holds. However, it is not in general true that a proposition P is intensionally equivalent to a compact proposition P'; and so, again, one must find some way of extending the definition to these other propositions.

We may do something analogous to what we already did in the case of minimal propositions. Say that the proposition Q is *comprehensive* if it is a set of $\boldsymbol{s}$-worlds for some comprehensive subject-matter **s** and that Q is a *comprehensive basis for* P if Q is a comprehensive proposition intensionally equivalent to P. Note that, in a W-space, a comprehensive basis Q for a given proposition P will always exist since we can let Q be the set of world-states in which P is true.

We may now extend the relation of partial content to all propositions as follows (G for Gemes):

$P \geq^{\mathrm{G}} Q$ if for any comprehensive basis P' for P there is a comprehensive basis Q' for Q for which $P' \geq Q'$.

Thus no matter how closely we approximate to P via a comprehensive basis we can approximate sufficiently closely to Q to ensure that the one proposition contains the other.

We may now establish analogues of lemma 30 and theorem 31:

Lemma 35 *Suppose that P is a compact proposition and P' a comprehensive basis for P. Then $P' \geq P$.*

Proof Given that P' is a comprehensive basis for P, it will consist of a set of $\boldsymbol{s}'$-worlds for some subject-matter $\boldsymbol{s}'$ comprehensive for P' and hence for P and, given that P is a compact proposition, it will consist of a set of $\boldsymbol{s}$-worlds for some subject-matter $\boldsymbol{s}$ compact for P; and so $\boldsymbol{s} \sqsubseteq \boldsymbol{s}'$. (a) Suppose first that $w' \in P'$. Since w' is consistent, it is compatible with some $\boldsymbol{s}$-world w and, since $\boldsymbol{s} \sqsubseteq \boldsymbol{s}'$ and w' is an $\boldsymbol{s}'$-world, $w \sqsubseteq w'$. But w either loosely verifies or loosely falsifies P and, since w' loosely verifies P, it is clear that w loosely verifies P and hence that $w \in P$. (b) Now suppose $w \in P$. Then w is compatible with some $\boldsymbol{s}'$-world w' and so $w \sqsubseteq w'$. But since w' either loosely verifies or loosely falsifies P and, since w loosely verifies P, it is clear that w loosely verifies P' and hence that $w \in P'$. □

Theorem 36 *For compact propositions P and Q, $P \geq Q$ iff $P \geq^{\mathrm{G}} Q$.*

Proof Suppose $P \geq Q$ for compact propositions P and Q. Take any comprehensive basis P' for P. Then $P' \geq P$ by the lemma. We let the comprehensive basis Q' for Q be Q itself. Given that $P \geq Q$, $P' \geq P \geq Q'$; and so $P \geq^{\mathrm{G}} Q$. Now suppose $P \geq^{\mathrm{G}} Q$. Given that P is compact, it is a comprehensive basis for P. So for some comprehensive basis Q' for Q, $P \geq Q'$. But $Q' \geq Q$ by the lemma; and so $P \geq Q$. □

These results (theorems 34 and 36) do not, of course, establish the adequacy of the proposed definition of $P \geq^{\mathrm{Y}} Q$ for cases in which P and Q are not both minimal or the adequacy of the proposed definition of $P \geq^{\mathrm{G}} Q$ for cases in which P and Q are not both compact. I suspect that there is no reasonable alternative to our treatment of these cases, but further work would be required to show that this was indeed so.

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